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Algebras of symmetric and block-symmetric functions on spaces of Lebesgue measurable functions

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In this work, we investigate algebras of symmetric and block-symmetric polynomials and analytic functions on complex Banach spaces of Lebesgue measurable functions for which the pth power of the absolute value is Lebesgue integrable, where $p \in [1, +\infty)$, and Lebesgue measurable essentially bounded functions on $[0, 1]$. We show that spectra of Fréchet algebras of block-symmetric entire functions of bounded type on these spaces consist only of point-evaluation functionals. Also we construct algebraic bases of algebras of continuous block-symmetric polynomials on these spaces. We generalize the above-mentioned results to a wide class of algebras of symmetric entire functions.

Key words and phrases: symmetric function, block-symmetric function, analytic function on Banach space, space of Lebesgue measurable functions, spectrum of algebra, algebraic basis.

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Introduction

Symmetric functions on Banach spaces were studied in [1–3,5–16,22,27]. In [26], it is shown that two Fréchet algebras of symmetric entire functions of bounded type on complex Banach spaces are isomorphic if semigroups of symmetries on underlying Banach spaces satisfy some natural conditions. In the current work, we continue the investigation of such isomorphisms of algebras.

Suppose we have two isomorphic topological algebras of functions and the spectrum of one of these algebras consists only of point-evaluation functionals. Does the spectrum of another algebra also consist only of point-evaluation functionals? In general case, the answer is "no" (see, e.g., [13]). But, as it is shown in the current work, if the isomorphism of algebras is generated by some bijection of underlying sets, the answer is "yes". We apply this result to Fréchet algebras of symmetric entire functions of bounded type on complex Banach spaces. In particular, we show that spectra of Fréchet algebras of block-symmetric entire functions of bounded type on complex Banach spaces of Lebesgue measurable functions for which the pth power of the absolute value is Lebesgue integrable, where $p \in [1, +\infty)$, and Lebesgue measurable essentially bounded functions on $[0, 1]$ consist only of point-evaluation functionals. Also we construct algebraic bases of algebras of continuous block-symmetric polynomials on these spaces. We generalize the above-mentioned results to a wide class of algebras of symmetric

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entire functions. Examples of such algebras are constructed.

Results of the work can be used in investigations of algebras of symmetric and weakly symmetric (see [25]) functions on Banach spaces.

1 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers. Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Let μ be the Lebesgue measure on $[0,1]$.

Let us define the function $\lambda_{[a,b]} : [a,b] \to [0,1]$ by

$$\lambda_{[a,b]}(t) = \frac{t-a}{b-a}. \tag{1}$$

Note that $\lambda_{[a,b]}$ is a bijection.

Symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f : A \to B$ is called S-symmetric if $f\big(s(a)\big) = f(a)$ for every $a \in A$ and $s \in S$.

The algebra $H_b(X)$. Let X be a complex Banach space. Let $H_b(X)$ be the Fréchet algebra of all entire functions $f : X \to \mathbb{C}$, which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets.

Let

$$\|f\|_r = \sup_{\|x\| \le r} \big|f(x)\big|$$

for $f \in H_b(X)$ and $r > 0$. The topology of $H_b(X)$ can be generated by an arbitrary set of norms $\big\{\|\cdot\|_r : r \in \Gamma\big\}$, where Γ is any unbounded subset of $(0, +\infty)$.

The algebras $H_{b,S}(X)$ and $\mathcal{P}_S(X)$. Let X be a complex Banach space. Let S be a set of operators on X. Let $H_{b,S}(X)$ be the subalgebra of all S-symmetric elements of $H_b(X)$. By [20, Lemma 3], $H_{b,S}(X)$ is closed in $H_b(X)$. So, $H_{b,S}(X)$ is a Fréchet algebra. Let $\mathcal{P}_S(X)$ be the subalgebra of $H_{b,S}(X)$ that consists of all S-symmetric continuous polynomials on X.

The group of bijections $\Xi_{[0,1]}$. Let $\Xi_{[0,1]}$ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure, i.e. for every Lebesgue measurable set $E \subset [0,1]$, both sets $\sigma(E)$ and $\sigma^{-1}(E)$ are Lebesgue measurable and $\mu\big(\sigma(E)\big) = \mu\left(\sigma^{-1}(E)\right) = \mu(E)$. Note that $\Xi_{[0,1]}$ is a group with respect to the operation of composition.

The group of bijections $\Xi^{(n)}_{[0,1]}$. Let $n \in \mathbb{N}$. Let $\Xi^{(n)}_{[0,1]}$ be the set of all bijections $\sigma \in \Xi_{[0,1]}$ such that

$$\sigma\left(t + 1/n\right) = \sigma(t) + 1/n$$

for every $t \in [0, 1 - 1/n]$. By [26, Proposition 2], $\Xi^{(n)}_{[0,1]}$ is a subgroup of $\Xi_{[0,1]}$.

The group of operators $S(\Xi, X^n)$. Let Ξ be an arbitrary subgroup of $\Xi_{[0,1]}$. Let X be an arbitrary linear space of equivalence classes with respect to the equivalence relation $x \sim y \Leftrightarrow x \overset{\text{a.e.}}{=} y$ of Lebesgue measurable functions on $[0,1]$ such that $x \circ \sigma$ belongs to X for every $x \in X$ and $\sigma \in \Xi$. Let X^n be the nth Cartesian power of X, where $n \in \mathbb{N}$. For $\sigma \in \Xi$, let the operator s_σ be defined by

$$s_\sigma : (x_1, \ldots, x_n) \in X^n \mapsto (x_1 \circ \sigma, \ldots, x_n \circ \sigma) \in X^n.$$

Let

$$S(\Xi, X^n) = \{s_\sigma : \sigma \in \Xi\}.$$

It can be verified that $S(\Xi, X^n)$ is a group of operators on X^n. If the context is clear, we shall write $S(\Xi)$ instead of $S(\Xi, X^n)$. Note that $\mathcal{S}\big(\Xi_{[0,1]}, X^n\big)$-symmetric functions on X^n are usually called symmetric and $\mathcal{S}\big(\Xi^{(n)}_{[0,1]}, X\big)$-symmetric functions on X are called n-block-symmetric.

The Cartesian power of $L_p[0,1]$. Let $L_p[0,1]$, where $p \in [1, +\infty)$, be the complex Banach space of all Lebesgue measurable functions $x : [0,1] \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable with norm

$$\|x\|_p = \left(\int_{[0,1]} |x(t)|^p \, dt \right)^{1/p}.$$

Let $\big(L_p[0,1]\big)^n$, where $n \in \mathbb{N}$, be the nth Cartesian power of $L_p[0,1]$ with norm

$$\|x\|_{p,n} = \Big(\sum_{s=1}^{n} \int_{[0,1]} |x_s(t)|^p \, dt \Big)^{1/p},$$

where $x = (x_1, \ldots, x_n) \in \big(L_p[0,1]\big)^n$.

The Cartesian power of $L_\infty[0,1]$. Let $L_\infty[0,1]$ be the complex Banach space of all Lebesgue measurable essentially bounded functions $x : [0,1] \to \mathbb{C}$ with norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t \in [0,1]} |x(t)|.$$

Let $\big(L_\infty[0,1]\big)^n$, where $n \in \mathbb{N}$, be the nth Cartesian power of $L_\infty[0,1]$ with norm

$$\|x\|_{\infty,n} = \max_{1 \le j \le n} \|x_j\|_\infty,$$

where $x = (x_1, \ldots, x_n) \in \big(L_\infty[0,1]\big)^n$.

Symmetric functions on Cartesian powers of $L_p[0,1]$ and $L_\infty[0,1]$. Let X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \ge 1$. By the definition of the group of operators $S\left(\Xi_{[0,1]}, X^n\right)$, a function f on X^n is $S\left(\Xi_{[0,1]}, X^n\right)$-symmetric if

$$f\big((x_1 \circ \sigma, \ldots, x_n \circ \sigma)\big) = f\big((x_1, \ldots, x_n)\big)$$

for every $(x_1, \ldots, x_n) \in X^n$ and $\sigma \in \Xi_{[0,1]}$. Note that $S(\Xi_{[0,1]}, X^n)$-symmetric functions on Cartesian powers of $L_p[0,1]$ and $L_\infty[0,1]$ were studied in works [17–21, 23, 24].

Let

$$M_{X,n} = \begin{cases} \{k \in \mathbb{Z}_+^n : 1 \le |k| \le p\}, & \text{if } X = L_p[0,1], \\ \{k \in \mathbb{Z}_+^n : |k| \ge 1\}, & \text{if } X = L_\infty[0,1], \end{cases} \tag{2}$$

where $|k| = k_1 + \ldots + k_n$ for $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$.

For every multi-index $k \in M_{X,n}$, let us define the mapping $R_{k,X^n} : X^n \to \mathbb{C}$ by

$$R_{k,X^n}(y) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s>0}}^{n} \big(y_s(t)\big)^{k_s}\, dt, \tag{3}$$

where $y = (y_1, \ldots, y_n) \in X^n$. Note that R_{k,X^n} is a symmetric continuous $|k|$-homogeneous polynomial.

By [17, Corollary 2.11] (for the case $X = L_p[0,1]$) and by [19, Corollary 3] (for the case $X = L_\infty[0,1]$), the following theorem holds.

Theorem 1. *Let X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \ge 1$. The set of polynomials $\{R_{k,X^n} : k \in M_{X,n}\}$, where R_{k,X^n} is defined by* (3) *and $M_{X,n}$ is defined by* (2)*, is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}(\Xi_{[0,1]})}(X^n)$.*

The isomorphism of Fréchet algebras of symmetric functions

Theorem 2 ([26, Theorem 2])**.** *Let X and Y be complex Banach spaces. Let S_1 and S_2 be semigroups of operators on X and Y respectively. Let $\iota : X \to Y$ be an isomorphism such that*

1) for every $x \in X$ and $s_1 \in S_1$, there exists $s_2 \in S_2$ such that $\iota\big(s_1(x)\big) = s_2\big(\iota(x)\big)$;

2) for every $y \in Y$ and $s_2 \in S_2$, there exists $s_1 \in S_1$ such that $\iota^{-1}\big(s_2(y)\big) = s_1\left(\iota^{-1}(y)\right)$.

Then the mapping

$$I : g \in H_{b,S_2}(Y) \mapsto g \circ \iota \in H_{b,S_1}(X) \tag{4}$$

is an isomorphism, i.e. I is a continuous linear multiplicative bijection.

2 Generating systems of algebras

Let A be a unital commutative algebra over some field $\mathbb{K}$. For every polynomial $Q : \mathbb{K}^n \to \mathbb{K}$ of the form

$$Q(z_1, \ldots, z_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1} \cdots z_n^{k_n}, \quad \alpha_{(k_1,\ldots,k_n)} \in \mathbb{K},$$

where Ω is some nonempty finite subset of $\mathbb{Z}_+^n$, let us define the mapping $Q_A : A^n \to A$ by

$$Q_A(a_1, \ldots, a_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} a_1^{k_1} \cdots a_n^{k_n}, \quad a_1, \ldots, a_n \in A, \tag{5}$$

(we consider the zeroth power a_j^0 of an element a_j to be the unit element of A).

Definition 1. *Let $a, a_1, \ldots, a_n \in A$. If there exists a polynomial $Q : \mathbb{K}^n \to \mathbb{K}$ such that $a = Q_A(a_1, \ldots, a_n)$, then a is called an* algebraic combination *of $a_1, \ldots, a_n$.*

Definition 2. *A nonempty set $G \subset A$ is called a* generating system *of A if every element of A can be represented as an algebraic combination of some elements of G. Furthermore, if every such a representation is unique, then G is called an* algebraic basis *of A.*

Definition 3. *A finite nonempty set $\{a_1, \ldots, a_n\} \subset A$ is called* algebraically independent *if the equality $Q_A(a_1, \ldots, a_n) = 0$ is possible only if the polynomial Q is identically equal to zero. An infinite set $A_0 \subset A$ is called* algebraically independent *if every its finite nonempty subset is algebraically independent.*

Evidently, every algebraic basis is algebraically independent. Furthermore, every algebraically independent generating system is an algebraic basis.

Proposition 1. *Let A_1 and A_2 be unital commutative algebras over a field $\mathbb{K}$. Let $h : A_1 \to A_2$ be a homomorphism.*

a) Let $a_1, \ldots, a_n \in A_1$. Let $Q : \mathbb{K}^n \to \mathbb{K}$ be a polynomial. Then

$$h\big(Q_{A_1}(a_1, \ldots, a_n)\big) = Q_{A_2}\big(h(a_1), \ldots, h(a_n)\big),$$

where Q_{A_1} and Q_{A_2} are defined by (5).

b) Let h be surjective. Let G be a generating system in A_1. Then $h(G)$ is a generating system in A_2.

c) Let h be injective. Let $C \subset A_2$ be an algebraically independent set such that $h^{-1}(C)$ is nonempty. Then $h^{-1}(C)$ is algebraically independent.

Proof. a) Using the linearity and the multiplicativity of h and taking into account (5), we obtain the result.

b) Let $b \in A_2$. Let us show that b can be represented as an algebraic combination of some elements of $h(G)$. Since h is surjective, the set $h^{-1}(b)$ is nonempty. Let $a \in h^{-1}(b)$. Since G is a generating system in A_1, it follows that there exist $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$ and a polynomial $Q : \mathbb{K}^n \to \mathbb{K}$ such that

$$a = Q_{A_1}(g_1, \ldots, g_n).$$

Then, by *a)*,

$$h(a) = Q_{A_2}\big(h(g_1), \ldots, h(g_n)\big).$$

Since $a \in h^{-1}(b)$, it follows that $h(a) = b$. Consequently,

$$b = Q_{A_2}\big(h(g_1), \ldots, h(g_n)\big).$$

So, b is an algebraic combination of elements of $h(G)$. Thus, $h(G)$ is a generating system in A_2.

c) Let us show that $h^{-1}(C)$ is algebraically independent, i.e. that every its finite nonempty subset is algebraically independent. Let $\{a_1, \ldots, a_n\} \subset h^{-1}(C)$. Let $Q : \mathbb{K}^n \to \mathbb{K}$ be a polynomial such that

$$Q_{A_1}(a_1, \ldots, a_n) = 0. \tag{6}$$

Let us show that Q is identically equal to zero. By *a)* and (6),

$$Q_{A_2}\big(h(a_1), \ldots, h(a_n)\big) = 0. \tag{7}$$

Since the set C is algebraically independent and $h(a_1), \ldots, h(a_n) \in C$, it follows that the set $\big\{h(a_1), \ldots, h(a_n)\big\}$ is algebraically independent. Therefore, the equality (7) is possible only if Q is identically equal to zero. So, $Q = 0$. Thus, the set $\{a_1, \ldots, a_n\}$ is algebraically independent. Hence, $h^{-1}(C)$ is algebraically independent. $\square$

Corollary 1. *Let A_1 and A_2 be unital commutative algebras over a field $\mathbb{K}$. Let $I : A_1 \to A_2$ be an isomorphism. Let B be an algebraic basis in A_1. Then $I(B)$ is an algebraic basis in A_2.*

Proof. Since B is an algebraic basis in A_1, it follows that B is a generating system in A_1 and B is algebraically independent. By the item *b)* of Proposition 1, since B is a generating system in A_1 and I is, in particular, a surjective homomorphism, it follows that $I(B)$ is a generating system in A_2. By the item *c)* of Proposition 1, where we set A_2, A_1, I^{-1} and B in place of A_1, A_2, h and C respectively, since B is algebraically independent, it follows that $I(B)$ is algebraically independent. So, $I(B)$ is an algebraically independent generating system in A_2, i.e. $I(B)$ is an algebraic basis in A_2. $\square$

3 Point-evaluation functionals on isomorphic algebras

Let us denote by $\mathcal{M}(A)$ the spectrum (the set of all nontrivial continuous linear multiplicative functionals) of a topological algebra A.

Lemma 1. *Let A_1 and A_2 be topological algebras over the same field. Let I be an isomorphism between A_1 and A_2. Then $\varphi \circ I \in \mathcal{M}(A_1)$ for every $\varphi \in \mathcal{M}(A_2)$.*

Proof. Let $\varphi \in \mathcal{M}(A_2)$. Let us show that $\varphi \circ I \in \mathcal{M}(A_1)$. Since both φ and I are linear, multiplicative and continuous, it follows that $\varphi \circ I$ is linear, multiplicative and continuous. Let us show that $\varphi \circ I$ is nontrivial. Suppose $\varphi \circ I$ is trivial, i.e.

$$(\varphi \circ I)(f) = 0 \tag{8}$$

for every $f \in A_1$. Let g be an arbitrary element of A_2. Let $f = I^{-1}(g)$. Then, by (8),

$$(\varphi \circ I)\left(I^{-1}(g)\right) = 0,$$

i.e. $\varphi(g) = 0$. Thus, $\varphi(g) = 0$ for every $g \in A_2$, which contradicts the nontriviality of φ. So, $\varphi \circ I$ is nontrivial. Thus, $\varphi \circ I \in \mathcal{M}(A_1)$. $\square$

Let $A(T)$ be a topological algebra of some functions on a nonempty set T. For $x \in X$, let $\delta_x(f) = f(x)$, where $f \in A(T)$. The mapping δ_x is called point-evaluation functional. Note that δ_x is linear and multiplicative.

Theorem 3. *Let T_1 and T_2 be nonempty sets. Let $\iota : T_1 \to T_2$ be a bijection. Let $A(T_1)$ and $B(T_2)$ be topological algebras of some functions on T_1 and T_2 respectively. Suppose the following conditions are satisfied:*

1) $g \circ \iota \in A(T_1)$ *for every* $g \in B(T_2)$;

2) the mapping

$$I : g \in B(T_2) \mapsto g \circ \iota \in A(T_1) \tag{9}$$

is an isomorphism;

3) the spectrum of the algebra $A(T_1)$ coincides with the set of point-evaluation functionals on some set $T_1^{(0)} \subset T_1$, i.e.

$$\mathcal{M}(A(T_1)) = \left\{\delta_x : x \in T_1^{(0)}\right\}.$$

Then

$$\mathcal{M}(B(T_2)) = \left\{\delta_{\iota(x)} : x \in T_1^{(0)}\right\}.$$

Proof. Let $\varphi \in \mathcal{M}\big(B(T_2)\big)$. Since I, defined by (9), is an isomorphism, it follows that I^{-1} is also an isomorphism. Then, by Lemma 1, the mapping $\varphi \circ I^{-1}$ belongs to $\mathcal{M}\big(A(T_1)\big)$. Consequently, taking into account the condition *3)*, there exists $x \in T_1^{(0)}$ such that

$$\varphi \circ I^{-1} = \delta_x. \tag{10}$$

Let us show that $\varphi = \delta_{\iota(x)}$. Let g be an arbitrary element of $B(T_2)$. Let

$$f = I(g). \tag{11}$$

Since $f \in A(T_1)$, by (10), we get

$$\left(\varphi \circ I^{-1}\right)(f) = \delta_x(f). \tag{12}$$

By (11), we have

$$\left(\varphi \circ I^{-1}\right)(f) = \left(\varphi \circ I^{-1}\right)\big(I(g)\big) = \varphi(g). \tag{13}$$

On the other hand, by (9) and (11), we obtain

$$\delta_x(f) = f(x) = I(g)(x) = (g \circ \iota)(x) = g\big(\iota(x)\big) = \delta_{\iota(x)}(g). \tag{14}$$

So, by (12), (13) and (14), we get $\varphi(g) = \delta_{\iota(x)}(g)$. Since the latter equality holds for every $g \in B(T_2)$, it follows that $\varphi = \delta_{\iota(x)}$. This completes the proof. $\square$

4 Isomorphisms of algebras of symmetric functions

Theorem 4. *Let X and Y be complex Banach spaces. Let S_1 and S_2 be semigroups of operators on X and Y respectively. Let $\iota : X \to Y$ be an isomorphism such that conditions 1) and 2) of Theorem 2 are satisfied. Let I be the isomorphism defined by* (4)*. Then*

- *a) the restriction of I to $\mathcal{P}_{S_2}(Y)$ is an isomorphism between algebras $\mathcal{P}_{S_2}(Y)$ and $\mathcal{P}_{S_1}(X)$;*
- *b) if $\mathcal{P}_{S_2}(Y)$ has some algebraic basis B, then $I(B)$ is an algebraic basis in $\mathcal{P}_{S_1}(X)$;*
- *c) if the spectrum of the algebra $H_{b,S_2}(Y)$ consists of point-evaluation functionals at points of some subset Y_0 of Y, then the spectrum of the algebra $H_{b,S_1}(X)$ consists of point-evaluation functionals at points of the set $\iota^{-1}(Y_0)$.*

Proof. *a)* Let us show that $I\big(\mathcal{P}_{S_2}(Y)\big) \subset \mathcal{P}_{S_1}(X)$. Let $P \in I\big(\mathcal{P}_{S_2}(Y)\big)$. Let us show that $P \in \mathcal{P}_{S_1}(X)$. Since $P \in I\big(\mathcal{P}_{S_2}(Y)\big)$, there exists $Q \in \mathcal{P}_{S_2}(Y)$ such that $P = I(Q)$. By (4), $I(Q) = Q \circ \iota$, i.e. $P = Q \circ \iota$. Therefore, since Q is a polynomial and ι is a linear mapping, it follows that P is a polynomial. Since $P \in H_{b,S_1}(X)$, it follows that P is continuous and S_1-symmetric. Thus, P is a continuous S_1-symmetric polynomial, i.e. $P \in \mathcal{P}_{S_1}(X)$. So, $I\big(\mathcal{P}_{S_2}(Y)\big) \subset \mathcal{P}_{S_1}(X)$.

Let us show that $\mathcal{P}_{S_1}(X) \subset I\big(\mathcal{P}_{S_2}(Y)\big)$. Let $P \in \mathcal{P}_{S_1}(X)$ and $Q = I^{-1}(P)$. Since $P = I(Q)$, by (4), we get $P = Q \circ \iota$. Therefore $Q = P \circ \iota^{-1}$. Consequently, since P is a polynomial and ι^{-1} is a linear mapping, it follows that Q is a polynomial. Since $Q \in H_{b,S_2}(Y)$, it follows that Q is continuous and S_2-symmetric. So, Q is a continuous S_2-symmetric polynomial, i.e. $Q \in \mathcal{P}_{S_2}(Y)$. Therefore $I(Q) \in I\big(\mathcal{P}_{S_2}(Y)\big)$, i.e. $P \in I\big(\mathcal{P}_{S_2}(Y)\big)$. Thus, $\mathcal{P}_{S_1}(X) \subset I\big(\mathcal{P}_{S_2}(Y)\big)$.

Hence, $I(\mathcal{P}_{S_2}(Y)) = \mathcal{P}_{S_1}(X)$. Consequently, taking into account that I is an isomorphism, the restriction of I to $\mathcal{P}_{S_2}(Y)$ is an isomorphism between algebras $\mathcal{P}_{S_2}(Y)$ and $\mathcal{P}_{S_1}(X)$.

b) Suppose the algebra $\mathcal{P}_{S_2}(Y)$ has some algebraic basis B. Then, by Corollary 1, taking into account *a)*, $I(B)$ is an algebraic basis in $\mathcal{P}_{S_1}(X)$.

c) Suppose the spectrum of the algebra $H_{b,S_2}(Y)$ consists of point-evaluation functionals at points of some subset Y_0 of Y. Let us substitute $Y, Y_0, X, H_{b,S_2}(Y), H_{b,S_1}(X), \iota^{-1}, I^{-1}$ instead of $T_1, T_1^{(0)}, T_2, A(T_1), B(T_2), \iota, I$ respectively into Theorem 3. Then the spectrum of the algebra $H_{b,S_1}(X)$ consists of point-evaluation functionals at points of the set $\iota^{-1}(Y_0)$. $\square$

Let us apply Theorem 4 to algebras of symmetric functions on spaces of Lebesgue measurable functions.

Theorem 5. *Let $n \in \mathbb{N}$ and let X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \in [1, +\infty)$. Then*

a) the set

$$\{G_{k,n,X} : k \in M_{X,n}\}, \tag{15}$$

where $M_{X,n}$ is defined by (2) *and $G_{k,n,X} : X \to \mathbb{C}$ is defined by*

$$G_{k,n,X}(x) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s>0}}^{n} \left(x\left(\frac{s-1+t}{n}\right)\right)^{k_s} dt, \tag{16}$$

is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ of all continuous n-block symmetric polynomials on X;

b) the spectrum of the algebra $H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ of all entire n-block symmetric functions of bounded type on X consists of point-evaluation functionals.

Proof. For $x = (x_1, \ldots, x_n) \in X^n$, let us define the function $\iota_{X,n}(x) : [0,1] \to \mathbb{C}$ by

$$\iota_{X,n}(x)(t) = \begin{cases} \left(x_j \circ \lambda_{[(j-1)/n, j/n]}\right)(t), & \text{if } t \in \left[(j-1)/n, j/n\right), \ j \in \{1, \ldots, n\}, \\ 0, & \text{if } t = 1, \end{cases} \tag{17}$$

where $\lambda_{[(j-1)/n,j/n]}$ is defined by (1). Let us define the mapping $\iota_{X,n} : X^n \to X$ by

$$\iota_{X,n} : x \in X^n \mapsto \iota_{X,n}(x) \in X, \tag{18}$$

where $\iota_{X,n}(x)$ is defined by (17). By [26, Proposition 5], the mapping $\iota_{X,n}$, defined by (18), is an isomorphism.

Let us substitute $X, X^n, \mathcal{S}(\Xi_{[0,1]}^{(n)}, X), \mathcal{S}(\Xi_{[0,1]}^{(1)}, X^n), \iota_{X,n}^{-1}$ instead of X, Y, S_1, S_2, ι into Theorem 4, respectively. By [26, Corollary 7] and [26, Corollary 8], the conditions *1)* and *2)* of Theorem 2, which are also required for Theorem 4, are satisfied. So, by Theorem 2, the mapping

$$I : g \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(1)}\right)}(X^n) \mapsto g \circ \iota_{X,n}^{-1} \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$$

is an isomorphism. By [26, Proposition 4], a function on X^n is $\mathcal{S}(\Xi_{[0,1]}, X^n)$-symmetric if and only if it is $\mathcal{S}(\Xi_{[0,1]}^{(1)}, X^n)$-symmetric. Therefore

$$H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(1)}\right)}(X^n) = H_{b,\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n).$$

So, in fact, I is an isomorphism between algebras $H_{b,\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n)$ and $H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$.

Let us prove the item *a)* of the current theorem. By Theorem 1, the set of polynomials

$$\{R_{k,X^n} : k \in M_{X,n}\},$$

where R_{k,X^n} is defined by (3) and $M_{X,n}$ is defined by (2), is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n)$. Therefore, by the item *b)* of Theorem 4, the set of polynomials

$$\{I\left(R_{k,X^n}\right) : k \in M_{X,n}\}$$

is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$. By [4, equalities (12) and (13)], we have

$$I(R_{k,X^n}) = G_{k,n,X}.$$

Thus, the set of polynomials $\{G_{k,n,X} : k \in M_{X,n}\}$ is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$.

Let us prove the item *b)* of the current theorem. By [20, Theorem 5] (for the case $X = L_p[0,1]$) and by [21, Theorem 5] (for the case $X = L_\infty[0,1]$) the spectrum of the algebra $H_{b,\mathcal{S}\left(\Xi_{[0,1]}\right)}(X^n)$ consists of point-evaluation functionals. Therefore, by the item *c)* of Theorem 4, the spectrum of the algebra $H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ consists of point-evaluation functionals. $\square$

Let $n \in \mathbb{N}$ and let X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \in [1,+\infty)$. Let $\tau : [0,1] \to [0,1]$ be such that the mapping

$$\iota_\tau : x \in X \mapsto x \circ \tau \in X \tag{19}$$

is an isomorphism. Let

$$\mathcal{S}_\tau = \left\{\iota_\tau^{-1} \circ s \circ \iota_\tau : s \in \mathcal{S}\left(\Xi_{[0,1]}^{(n)}, X\right)\right\}. \tag{20}$$

It can be checked that $\mathcal{S}_\tau$ is a group of operators on X. Let us establish some properties of the algebra $H_{b,\mathcal{S}_\tau}(X)$ of all $\mathcal{S}_\tau$-symmetric entire functions of bounded type on X and the algebra $\mathcal{P}_{\mathcal{S}_\tau}(X)$ of all $\mathcal{S}_\tau$-symmetric continuous polynomials on X.

Theorem 6. *Let $n \in \mathbb{N}$ and let X be equal to $L_p[0,1]$ or $L_\infty[0,1]$, where $p \in [1,+\infty)$. Let $\tau : [0,1] \to [0,1]$ be such that the mapping ι_τ, defined by* (19)*, is an isomorphism. Then*

a) the mapping

$$I_\tau : g \in H_{b,\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X) \mapsto g \circ \iota_\tau \in H_{b,\mathcal{S}_\tau}(X) \tag{21}$$

is an isomorphism, i.e. I_τ is a continuous linear multiplicative bijection;

b) the restriction of I_τ to $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ is an isomorphism between algebras $\mathcal{P}_{\mathcal{S}\left(\Xi_{[0,1]}^{(n)}\right)}(X)$ and $\mathcal{P}_{\mathcal{S}_\tau}(X)$;

c) the set of polynomials

$$\{x \mapsto G_{k,n,X}(x \circ \tau) : k \in M_{X,n}\}, \tag{22}$$

where the set $M_{X,n}$ is defined by (2) *and polynomials $G_{k,n,X}$ are defined by* (16)*, is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}_\tau}(X)$;*

d) the spectrum of the algebra $H_{b,\mathcal{S}_\tau}(X)$ consists of point-evaluation functionals.

Proof. Let us substitute $X, X, \mathcal{S}_\tau, \mathcal{S}\big(\Xi^{(n)}_{[0,1]}, X\big), \iota_\tau$ instead of X, Y, S_1, S_2, ι, respectively, into Theorem 2 and Theorem 4. Let us check the condition *1)* of Theorem 2. Let $x \in X$ and $s_1 \in \mathcal{S}_\tau$. Then, by (20), there exists $s \in \mathcal{S}\big(\Xi^{(n)}_{[0,1]}, X\big)$ such that $s_1 = \iota_\tau^{-1} \circ s \circ \iota_\tau$. Let $s_2 = s$. Then $\iota_\tau\big(s_1(x)\big) = s_2\big(\iota_\tau(x)\big)$. Thus, the condition *1)* of Theorem 2 is satisfied.

Let us check the condition *2)* of Theorem 2. Take $x \in X$ and $s_2 \in \mathcal{S}\big(\Xi^{(n)}_{[0,1]}, X\big)$. Let $s_1 = \iota_\tau^{-1} \circ s_2 \circ \iota_\tau$. By (20), we have $s_1 \in \mathcal{S}_\tau$. Note that $s_1 \circ \iota_\tau^{-1} = \iota_\tau^{-1} \circ s_2$. Therefore $s_1\left(\iota_\tau^{-1}(x)\right) = \iota_\tau^{-1}\big(s_2(x)\big)$. Thus, the condition *2)* of Theorem 2 is satisfied.

By Theorem 2, the mapping I_τ, defined by (21), is an isomorphism. So, the item *a)* of the current theorem holds.

The item *a)* of Theorem 4 implies the item *b)* of the current theorem.

By the item *a)* of Theorem 5, the set (15) is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}\left(\Xi^{(n)}_{[0,1]}\right)}(X)$. Consequently, by the item *b)* of Theorem 4, the set $\big\{I_\tau(G_{k,n,X}) : k \in M_{X,n}\big\}$, where the set $M_{X,n}$ is defined by (2) and polynomials $G_{k,n,X}$ are defined by (16), is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}_\tau}(X)$. By (21) and (19), we have

$$I_\tau(G_{k,n,X}) = G_{k,n,X} \circ \iota_\tau.$$

Therefore, by (19), we get

$$I_\tau(G_{k,n,X})(x) = (G_{k,n,X} \circ \iota_\tau)(x) = G_{k,n,X}(x \circ \tau)$$

for every $x \in X$. So, the set (22) is an algebraic basis of the algebra $\mathcal{P}_{\mathcal{S}_\tau}(X)$. This completes the proof of the item *c)* of the current theorem.

The item *c)* of Theorem 4 and the item *b)* of Theorem 5 imply the item *d)* of the current theorem. ◻

5 Isomorphisms of $L_\infty[0,1]$

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For an arbitrary set $A \subset [0,1]$, let $1_A : [0,1] \to \mathbb{K}$ be defined by

$$1_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \in [0,1] \setminus A. \end{cases}$$

Proposition 2. *Let $\tau : [0,1] \to [0,1]$. The following conditions are equivalent:*

1) *for every Lebesgue measurable function $x : [0,1] \to \mathbb{K}$, the function $x \circ \tau$ is Lebesgue measurable;*

2) *for every Lebesgue measurable set $A \subset [0,1]$, the set $\tau^{-1}(A)$ is Lebesgue measurable.*

Proof. Suppose the condition *1)* holds. Let $A \subset [0,1]$ be a Lebesgue measurable set. Let us show that the set $\tau^{-1}(A)$ is Lebesgue measurable. Let $x = 1_A$. Let $B \subset \mathbb{K}$ be an arbitrary Borel set that contains 1 and does not contain 0. By *1)*, $x \circ \tau$ is a Lebesgue measurable function. Therefore, the set $(x \circ \tau)^{-1}(B)$ is Lebesgue measurable. Note that

$$(x \circ \tau)^{-1}(B) = \tau^{-1}\left(x^{-1}(B)\right) = \tau^{-1}\left(1_A^{-1}(B)\right) = \tau^{-1}(A).$$

Thus, $\tau^{-1}(A)$ is a Lebesgue measurable set. So, the condition *2)* holds.

Suppose the condition *2)* holds. Let $x : [0,1] \to \mathbb{K}$ be a Lebesgue measurable function. Let us show that $x \circ \tau$ is a Lebesgue measurable function, i.e. the inverse image with respect to $x \circ \tau$ of every Borel set is a Lebesgue measurable set. Let $B \subset \mathbb{K}$ be a Borel set. Since x is a Lebesgue measurable function, the set $x^{-1}(B)$ is Lebesgue measurable. Therefore, taking into account the condition *2)*, the set $\tau^{-1}\left(x^{-1}(B)\right)$ is Lebesgue measurable. Consequently, taking into account the equality

$$(x \circ \tau)^{-1}(B) = \tau^{-1}\left(x^{-1}(B)\right),$$

the set $(x \circ \tau)^{-1}(B)$ is Lebesgue measurable. Thus, $x \circ \tau$ is a Lebesgue measurable function. So, the condition *1)* holds. □

Proposition 3. *Let $\tau : [0,1] \to [0,1]$. The following conditions are equivalent:*

1) $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$ for every Lebesgue measurable functions $x, y : [0,1] \to \mathbb{K}$ such that $x \overset{\text{a.e.}}{=} y$;

2) for every null set $N \subset [0,1]$, the set $\tau^{-1}(N)$ is a null set.

Proof. Suppose the condition *1)* holds. Let $N \subset [0,1]$ be a null set. Let $M = \tau^{-1}(N)$. Let us show that M is a null set. Let $x = 1_N$ and $y = 0$. Evidently, both x and y are Lebesgue measurable functions and, since N is a null set, $1_N \overset{\text{a.e.}}{=} 0$, i.e. $x \overset{\text{a.e.}}{=} y$. Therefore, by *1)*, we get $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$. Consequently, taking into account the equalities

$$x \circ \tau = 1_N \circ \tau = 1_M \quad \text{and} \quad y \circ \tau = 0 \circ \tau = 0,$$

we have $1_M \overset{\text{a.e.}}{=} 0$. Therefore M is a null set. Thus, the condition *2)* holds.

Suppose the condition *2)* holds. Let $x, y : [0,1] \to \mathbb{K}$ be Lebesgue measurable functions such that $x \overset{\text{a.e.}}{=} y$. Let us show that $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$, i.e.

$$M = \big\{t \in [0,1] : (x \circ \tau)(t) \neq (y \circ \tau)(t)\big\}$$

is a null set. Let

$$N = \big\{t \in [0,1] : x(t) \neq y(t)\big\}.$$

Since $x \overset{\text{a.e.}}{=} y$, it follows that N is a null set. Therefore, by the condition *2)*, $\tau^{-1}(N)$ is a null set. Let us show that $M \subset \tau^{-1}(N)$. Let $t \in M$. Then $(x \circ \tau)(t) \neq (y \circ \tau)(t)$, i.e. $x\big(\tau(t)\big) \neq y\big(\tau(t)\big)$. Therefore $\tau(t) \in N$. Consequently, $t \in \tau^{-1}(N)$. Thus, $M \subset \tau^{-1}(N)$. Consequently, M is a null set. So, $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$. Thus, the condition *1)* holds. □

Proposition 4. *Let $\tau : [0,1] \to [0,1]$ be such that $\tau(E)$ is a Lebesgue measurable set and $\mu\big(\tau(E)\big) = 1$ for every Lebesgue measurable set $E \subset [0,1]$ such that $\mu(E) = 1$. Let $x, y : [0,1] \to \mathbb{K}$ be some Lebesgue measurable functions such that $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$. Then $x \overset{\text{a.e.}}{=} y$.*

Proof. Let $x, y : [0,1] \to \mathbb{K}$ be Lebesgue measurable functions such that $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$. Let us show that $x \overset{\text{a.e.}}{=} y$. Let

$$E = \big\{t \in [0,1] : (x \circ \tau)(t) = (y \circ \tau)(t)\big\}. \tag{23}$$

Since $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$, it follows that the set E is Lebesgue measurable and $\mu(E) = 1$. Therefore, by the condition of the proposition, the set $\tau(E)$ is Lebesgue measurable and $\mu\big(\tau(E)\big) = 1$. By (23), $x\big(\tau(t)\big) = y\big(\tau(t)\big)$ for every $t \in E$. Therefore $x(\theta) = y(\theta)$ for every $\theta \in \tau(E)$. Consequently, taking into account the equality $\mu\big(\tau(E)\big) = 1$, we get $x \overset{\text{a.e.}}{=} y$. This completes the proof. □

Proposition 5. *Let $\tau : [0,1] \to [0,1]$ be such that the following conditions hold:*

1) *there exists a Lebesgue measurable set $E \subset [0,1]$ such that $\mu(E) = 1$ and the restriction of τ to E is injective;*

2) *for every Lebesgue measurable set $A \subset [0,1]$, the set $\tau(A)$ is Lebesgue measurable.*

Then for every Lebesgue measurable function $y : [0,1] \to \mathbb{K}$ there exists a Lebesgue measurable function $x : [0,1] \to \mathbb{K}$ such that $y \overset{\text{a.e.}}{=} x \circ \tau$.

Proof. Let $y : [0,1] \to \mathbb{K}$ be an arbitrary Lebesgue measurable function. Let us construct a Lebesgue measurable function $x : [0,1] \to \mathbb{K}$ such that $y \overset{\text{a.e.}}{=} x \circ \tau$. By the condition *1)*, the restriction of τ to E is injective. Consequently, for every $t \in \tau(E)$, the set $\tau^{-1}(t)$ contains exactly one element. Therefore the function

$$x(t) = \begin{cases} \left(y \circ \tau^{-1}\right)(t), & \text{if } t \in \tau(E), \\ 0, & \text{if } t \in [0,1] \setminus \tau(E) \end{cases}$$

is well defined.

Let us show that the function x is Lebesgue measurable. Let $B \subset \mathbb{K}$ be an arbitrary Borel set. Let us show that $x^{-1}(B)$ is a Lebesgue measurable set. Note that

$$x^{-1}(B) = \begin{cases} \tau\left(y^{-1}(B)\right), & \text{if } 0 \notin B, \\ \tau\left(y^{-1}(B)\right) \cup \left([0,1] \setminus \tau(E)\right), & \text{if } 0 \in B. \end{cases}$$

Consequently, it is enough to show that both sets $\tau\big(y^{-1}(B)\big)$ and $[0,1] \setminus \tau(E)$ are Lebesgue measurable. Since y is a Lebesgue measurable function, $y^{-1}(B)$ is a Lebesgue measurable set. Consequently, taking into account the condition *2)*, the set $\tau\big(y^{-1}(B)\big)$ is Lebesgue measurable. Since E is Lebesgue measurable, by the condition *2)*, the set $\tau(E)$ is Lebesgue measurable. Therefore, the set $[0,1] \setminus \tau(E)$ is Lebesgue measurable. So, $x^{-1}(B)$ is Lebesgue measurable. Thus, the function x is Lebesgue measurable.

Let us show that $y \overset{\text{a.e.}}{=} x \circ \tau$. For every $t \in E$,

$$(x \circ \tau)(t) = x\big(\tau(t)\big) = \left(y \circ \tau^{-1}\right)\big(\tau(t)\big) = y(t).$$

Thus, $y(t)=(x \circ \tau)(t)$ for every $t \in E$. Consequently, taking into account that $\mu(E) = 1$, we have $y \overset{\text{a.e.}}{=} x \circ \tau$. This completes the proof. □

Theorem 7. *Let $\tau : [0,1] \to [0,1]$ be such that*

1) *for every Lebesgue measurable set $A \subset [0,1]$, the set $\tau^{-1}(A)$ is Lebesgue measurable;*

2) *for every Lebesgue measurable set $A \subset [0,1]$, the set $\tau(A)$ is Lebesgue measurable;*

3) *for every null set $N \subset [0,1]$, the set $\tau^{-1}(N)$ is a null set;*

4) *for every Lebesgue measurable set $E \subset [0,1]$ such that $\mu(E) = 1$, the set $\tau(E)$ is Lebesgue measurable and $\mu\big(\tau(E)\big) = 1$;*

5) *there exists a Lebesgue measurable set $E \subset [0,1]$ such that $\mu(E) = 1$ and the restriction of τ to E is injective.*

Then the mapping

$$\iota_\tau : x \in L_\infty[0,1] \mapsto x \circ \tau \in L_\infty[0,1] \tag{24}$$

is an isomorphism.

Proof. Let us show that the mapping ι_τ, defined by (24), is well defined.

By the condition *3)* and by Proposition 3, $x \circ \tau \overset{\text{a.e.}}{=} y \circ \tau$ for every Lebesgue measurable functions x, y on $[0,1]$ such that $x \overset{\text{a.e.}}{=} y$. So, for equivalent functions x and y, functions $x \circ \tau$ and $y \circ \tau$ are equivalent. Thus, the result of the action of ι_τ to some class of equivalence that belongs to $L_\infty[0,1]$ does not depend on the choice of the representative of the class.

By the condition *1)* and by Proposition 2, for every Lebesgue measurable function x on $[0,1]$, the function $x \circ \tau$ is Lebesgue measurable. Thus, $\iota_\tau(x)$ is some class of equivalence consisting of Lebesgue measurable functions for every $x \in L_\infty[0,1]$. Let us show that $\iota_\tau(x) \in L_\infty[0,1]$. Note that

$$\operatorname*{ess\,sup}_{t\in[0,1]} \big|(x\circ\tau)(t)\big| = \operatorname*{ess\,sup}_{t\in\tau\left([0,1]\right)} \big|x(t)\big| \le \operatorname*{ess\,sup}_{t\in[0,1]} \big|x(t)\big|$$

for every $x \in L_\infty[0,1]$. Therefore

$$\big\|\iota_\tau(x)\big\|_\infty \le \|x\|_\infty \tag{25}$$

for every $x \in L_\infty[0,1]$. Thus, $\iota_\tau(x) \in L_\infty[0,1]$ for every $x \in L_\infty[0,1]$. So, the mapping ι_τ is well defined.

By the condition *4)* and by Proposition 4, the mapping ι_τ is injective.

By conditions *2)* and *5)* and by Proposition 5, the mapping ι_τ is surjective.

It can be checked that ι_τ is linear. Consequently, taking into account (25), ι_τ is continuous. So, ι_τ is a continuous linear bijection. Therefore, by the bounded inverse theorem, ι_τ^{-1} is continuous. Thus, the mapping ι_τ is an isomorphism. □

Let us denote by $\mathcal{T}$ the set of all the mappings $\tau : [0,1] \to [0,1]$ that satisfy all conditions of Theorem 7.

Corollary 2. *Let $n \in \mathbb{N}$, $X = L_\infty[0,1]$ and $\tau \in \mathcal{T}$. Then items a) – d) of Theorem 6 hold.*

Proof. By Theorem 7, the mapping

$$\iota_\tau : x \in L_\infty[0,1] \mapsto x \circ \tau \in L_\infty[0,1]$$

is an isomorphism. Therefore, conditions of Theorem 6 are satisfied. Consequently, items *a) – d)* of Theorem 6 hold. □

Let us construct some examples.

Example 1. *Let $n \in \mathbb{N}$ and $\theta_1, \ldots, \theta_n \in \mathcal{T}$. Let $\tau_{\theta_1,\ldots,\theta_n} : [0,1] \to [0,1]$ be defined by*

$$\tau_{\theta_1,\ldots,\theta_n}(t) = \begin{cases} \left(\lambda^{-1}_{[(j-1)/n,j/n]} \circ \theta_j \circ \lambda_{[(j-1)/n,j/n]}\right)(t), & \text{if } t \in \big[(j-1)/n, j/n\big), \ \ j \in \{1,\ldots,n\}, \\ 1, & \text{if } t = 1, \end{cases}$$

where $\lambda_{[(j-1)/n,j/n]}$ is defined by (1)*. It can be checked that $\tau_{\theta_1,\ldots,\theta_n} \in \mathcal{T}$. Therefore, by Corollary 2, in the case $\tau = \tau_{\theta_1,\ldots,\theta_n}$, items a) – d) of Theorem 6 hold. Note that*

$$G_{k,n,L_\infty[0,1]}(x \circ \tau_{\theta_1,\ldots,\theta_n}) = \int_{[0,1]} \prod_{\substack{s=1\\ k_s>0}}^{n} \left(x\left(\frac{s-1+\theta_s(t)}{n}\right)\right)^{k_s} dt,$$

for every $x \in L_\infty[0,1]$, *where* $G_{k,n,L_\infty[0,1]}$ *is defined by* (16). *Therefore, elements of the algebraic basis* (22) *have the form*

$$x \in L_\infty[0,1] \mapsto \int_{[0,1]} \prod_{\substack{s=1 \\ k_s>0}}^{n} \left(x\left(\frac{s-1+\theta_s(t)}{n} \right) \right)^{k_s} dt,$$

where $(k_1,\ldots,k_n) \in M_{X,n}$, *i.e. taking into account* (2), $(k_1,\ldots,k_n) \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}$.

Consider two specific examples.

a) Let $n=2$, $\theta_1(t)=t^\alpha$, $\theta_2(t)=t^\beta$, *where* $\alpha,\beta>0$. *Then elements of the algebraic basis* (22) *have the form*

$$x \in L_\infty[0,1] \mapsto \int_{[0,1]} \left(x\left(t^\alpha/2\right)\right)^{k_1} \left(x\left(1/2+t^\beta/2\right)\right)^{k_2} dt,$$

where $(k_1,k_2) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$.

b) Let $n=2$, $\theta_1(t)=t$, $\theta_2(t)=1-t$. *Then elements of the algebraic basis* (22) *have the form*

$$x \in L_\infty[0,1] \mapsto \int_{[0,1]} \left(x\left(t/2\right)\right)^{k_1} \left(x\left(1-t/2\right)\right)^{k_2} dt,$$

where $(k_1,k_2) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. *Note that*

$$\int_{[0,1]} \left(x\left(t/2\right)\right)^{k_1} \left(x\left(1-t/2\right)\right)^{k_2} dt = \int_{[0,1]} \left(x(t)\right)^{k_1} \left(x(1-t)\right)^{k_2} dt.$$

Thus, elements of the algebraic basis (22) *have the form*

$$x \in L_\infty[0,1] \mapsto \int_{[0,1]} \left(x(t)\right)^{k_1} \left(x(1-t)\right)^{k_2} dt,$$

where $(k_1,k_2) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$.

Example 2. *Let* $n \in \mathbb{N}$. *Let* $E_1,\ldots,E_n \subset [0,1]$ *be Lebesgue measurable sets such that* $\mu(E_j \cap E_k) = 0$ *if* $j \neq k$. *Then, by [9, Proposition 2.2], there exists* $\sigma_{E_1,\ldots,E_n} \in \Xi_{[0,1]}$ *such that* $1_{E_m} = 1_{[b_{m-1},b_m]} \circ \sigma_{E_1,\ldots,E_n}$ *for every* $m \in \{1,\ldots,n\}$ *almost everywhere on* $[0,1]$, *where* $b_0=0$ *and* $b_k = \sum_{j=1}^k \mu(E_j)$ *for* $k \in \{1,\ldots,n\}$. *Let* $\tau_{E_1,\ldots,E_n} : [0,1] \to [0,1]$ *be defined by*

$$\tau_{E_1,\ldots,E_n}(t) = \begin{cases} \left(\sigma^{-1}_{E_1,\ldots,E_n} \circ \lambda^{-1}_{[b_{j-1},b_j]} \circ \lambda_{[(j-1)/n,j/n]}\right)(t), & \text{if } t \in \left[(j-1)/n, j/n\right),\ j \in \{1,\ldots,n\}, \\ 1, & \text{if } t=1, \end{cases}$$

where $\lambda_{[(j-1)/n,j/n]}$ *is defined by* (1). *It can be checked that* $\tau_{E_1,\ldots,E_n} \in \mathcal{T}$. *Therefore, by Corollary 2, in the case* $\tau = \tau_{E_1,\ldots,E_n}$, *items a)–d) of Theorem 6 hold.*

Consider some specific example.

Let $n=1$. *Let* $E \subset [0,1]$ *be a Lebesgue measurable set such that* $\mu(E)>0$. *Then elements of the algebraic basis* (22) *have the form*

$$x \in L_\infty[0,1] \mapsto \frac{1}{\mu(E)} \int_E \left(x(t)\right)^k dt,$$

where $k \in \mathbb{N}$.

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Василишин Т.В. *Алгебри симетричних і блочно-симетричних функцій на просторах вимірних за Лебегом функцій* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 174–189.

У даній роботі досліджуються алгебри симетричних і блочно-симетричних поліномів і аналітичних функцій на комплексних банахових просторах вимірних за Лебегом функцій, для яких p-тий степінь абсолютного значення є інтегровним за Лебегом, де $p \in [1,+\infty)$, і вимірних за Лебегом суттєво обмежених функцій на відрізку $[0,1]$. Показано, що спектри алгебр Фреше блочно-симетричних цілих функцій обмеженого типу на цих просторах складаються виключно із функціоналів обчислення значень в точках. Також побудовано алгебраїчні базиси алгебр неперервних блочно-симетричних поліномів на цих просторах. Згадані результати узагальнено на широкий клас алгебр симетричних цілих функцій.

Ключові слова і фрази: симетрична функція, блочно-симетрична функція, аналітична функція на банаховому просторі, простір вимірних за Лебегом функцій, спектр алгебри, алгебраїчний базис.