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Discontinuous strongly separately continuous functions of several variables and near coherence of two P-filters

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We consider a notion of near coherence of n P-filters and show that the near coherence of any n P-filters is equivalent to the near coherence of any two P-filters. For any filter u on $\mathbb{N}$ by $\mathbb{N}_u$ we denote the space $\mathbb{N} \cup \{u\}$, in which all points from $\mathbb{N}$ are isolated and sets $A \cup \{u\}$, $A \in u$, are neighborhoods of u. In the article, the concept of strongly separately finite sets was introduced. For $X = \mathbb{N}_{u_1} \times \cdots \times \mathbb{N}_{u_n}$ we prove that the existence of a strongly separately continuous function $f : X \to \mathbb{R}$ with one-point set $\{(u_1, \ldots, u_n)\}$ of discontinuity implies the existence of a strongly separately finite set $E \subseteq X$ such that the characteristic function $\chi|_E$ is discontinuous at $(u_1, \ldots, u_n)$. Using this fact we proved that the existence of a strongly separately continuous function $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ on the product of arbitrary completely regular spaces X_k with an one-point set $\{(x_1, \ldots, x_n)\}$ of points of discontinuity, where x_k is non-isolated G_δ-point in X_k, is equivalent to near coherence of P-filters.

Key words and phrases: separately continuous function, strongly separately continuous function, P-filter, inverse problem, one-point discontinuity.

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Introduction

Investigations of the discontinuity point sets of separately continuous functions of two or many variables (i.e. functions that are continuous with respect to each variable) were started in R. Baire's dissertation [1]. These investigations have been continued and developed by many mathematicians (see [2] the literature given there). The question on characterization of the discontinuity point sets of separately continuous functions defined on the product of two compact spaces naturally arises in connection with well-known Namioka theorem [3]. This question was formulated by Z. Piotrowski [4] and it is still open.

Let us remark that a complete description of the discontinuity point sets of separately continuous functions was obtained only in the following two cases: for separately continuous functions on the product of n metrizable spaces [5], and for separately continuous functions on the product of n spaces, each of which is the product of separable metrizable spaces [6].

Investigations of separately continuous functions and their analogs with one-point set of points of discontinuity are of particular interest (see [2, 7–11]). In particular, necessity and sufficiency conditions for the existence of separately continuous function on the product of two compact spaces with one-point set of points of discontinuity were obtained in [8]. In [9], this result was generalized for functions of several variables.

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On the other hand, it was proved in [11] that the existence of separately continuous functions with given one-point set of points of discontinuity of G_δ type is closely related to the properties of P-filter, and the answer to this question is independent of ZFC (abbreviation of Zermelo-Fraenkel set theory with the Axiom of Choice).

Theorem ([11, Theorem 4])**.** *The following statements are equivalent.*

(1) For any completely regular spaces X, Y with non-isolated G_δ-points $a \in X$, $b \in Y$ there are continuous maps $g : X \to \mathbb{R}$, $h : Y \to \mathbb{R}$ and a function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the function $f = \varphi \circ (g \times h) : X \times Y \to \mathbb{R}$ is separately continuous, discontinuous at (a, b) and continuous at other points of $X \times Y$.

(2) Any two free P-filters on a countable set are near coherent.

In this paper, we generalize this result for the case of strongly separately continuous function of several variables on the product of completely regular spaces and show that for any number of corresponding spaces the existence of respective strongly separately continuous function is equivalent to the statement that any two P-filters are near coherent.

In Section 1, we give the definition of P-filters, the definition of the near coherence of n P-filters and the definition of $\mathcal{F}$. We prove some additional facts about filters. The main result of the section is the fact that the near coherence of any n P-filters from $\mathcal{F}$ is equivalent to the near coherence of any two P-filter from $\mathcal{F}$.

In Section 2, we give the definition of strongly separately continuous functions and the definition of strongly separately finite sets. We prove that for n P-filters $u_1, \dots, u_n$ from $\mathcal{F}$ the existence of a strongly separately continuous function $f : \prod_{i=1}^n \mathbb{N}_{u_i} \to \mathbb{R}$ with one-point set $\{(u_1, \dots, u_n)\}$ of discontinuity implies the existence of a strongly separately finite set $E \subseteq \prod_{i=1}^n \mathbb{N}_{u_i}$ such that the characteristic function $\chi|_E$ is discontinuous at $(u_1, \dots, u_n)$. Using that fact, we prove that for any n P-filters from $\mathcal{F}$, that are not near coherent, every strongly separately continuous function $f : \prod_{i=1}^n \mathbb{N}_{u_i} \to \mathbb{R}$ is continuous.

Finally, in Section 3, we prove that if any two P-filters from $\mathcal{F}$ are near coherent, then for any completely regular spaces $X_1, \dots, X_n$ with non-isolated G_δ-point x_{i0} in the corresponding space X_i there exists a strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ such that $D(f) = \{(x_{10}, \dots, x_{n0})\}$. Also, we give the main result of the article, which states the equivalence of the existing of a strongly separately continuous function with one-point discontinuity for n completely regular spaces with non-isolated G_δ-point and the fact that any two P-filters from $\mathcal{F}$ are near coherent.

1 Filters from $\mathcal{F}$ and their properties

A non-empty system $\mathcal{A}$ of non-empty subsets of S is called a *filter* on S if the following conditions are true:

1) $A_1 \cap A_2 \in \mathcal{A}$ for any $A_1, A_2 \in \mathcal{A}$;

2) if $A \in \mathcal{A}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{A}$.

A filter $\mathcal{A}$ on a set S is called a *P-filter*, if for any sequence $(A_m)_{m=1}^\infty$ of sets $A_m \in \mathcal{A}$ there exists a set $A \in \mathcal{A}$ such that the set $A \setminus A_m$ is finite for any $m \in \mathbb{N}$.

The filters $\mathcal{A}$ and $\mathcal{B}$ on a set S are called *near coherent*, if there exists a function $\varphi : S \to S$ such that a set $\varphi^{-1}(s)$ is finite for every $s \in S$ and $\varphi(A) \cap \varphi(B) \neq \varnothing$ for any $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Note that the statement "any two P-filters on countable set are near coherent" was abbreviated in [11] as NCPF (near coherence of P-filters). Moreover, it was shown in [11, Theorem 5] that NCPF is independent of ZFC.

We denote $\mathcal{F}$ as a collection of all filters x on $\mathbb{N}$ such that for any $A, B \subseteq \mathbb{N}$ with finite symmetric difference $A \triangle B$, if $A \in x$, then $B \in x$. It is clear that the filter x on the set $\mathbb{N}$ belongs to $\mathcal{F}$ if and only if

$$\{k \in \mathbb{N} : k \geq m\} \in x$$

for every $m \in \mathbb{N}$.

For every $x \in \mathcal{F}$, by $\mathbb{N}_x$ we denote the space $\mathbb{N} \cup \{x\}$ in which all points $m \in \mathbb{N}$ are isolated and a set $A \cup \{x\}$, where $A \subseteq \mathbb{N}$, is a neighborhood of the point x in $\mathbb{N}_x$ if and only if $A \in x$.

Let us prove some properties of filters from $\mathcal{F}$.

Lemma 1. *Let x be a P-filter from $\mathcal{F}$, $\varphi : \mathbb{N} \to \mathbb{N}$ be such that for any $m \in \mathbb{N}$ a set $\varphi^{-1}(m)$ is finite. Then $y = \{B \subseteq \mathbb{N} : \text{ there exists a set } A \in x \text{ such that } \varphi(A) \subseteq B\}$ is a P-filter from $\mathcal{F}$.*

Proof. First we show that y is a filter.

1) Let $B_1, B_2 \in y$. We show that $B_1 \cap B_2 \in y$. According to the construction of y, there exist sets $A_1, A_2 \in x$ such that $\varphi(A_1) \subseteq B_1, \varphi(A_2) \subseteq B_2$. Since x is a filter, $A_1 \cap A_2 \in x$ and

$$\varphi(A_1 \cap A_2) \subseteq \varphi(A_1) \cap \varphi(A_2) \subseteq B_1 \cap B_2.$$

Hence, $B_1 \cap B_2 \in y$.

2) Let $B_1 \in y$ and $B_1 \subseteq B$. We show that $B \in y$. Since $B_1 \in y$, there exists a set $A \in x$ such that $\varphi(A) \subseteq B_1$. Moreover, since $B_1 \subseteq A$, $\varphi(A) \subseteq B$. Therefore $B \in y$. Hence, y is a filter.

Now we show that y is a P-filter. Let $(B_m)_{m=1}^{\infty}$ be a sequence of sets from y. Then for any $m \in \mathbb{N}$ there exists a set $A_m \in x$ such that $\varphi(A_m) \subseteq B_m$. Since x is a P-filter, there exists a set $A \in x$ such that for any $m \in \mathbb{N}$ a set $A \setminus A_m$ is finite. Put $B = \varphi(A)$. It is obviously that $B \in y$. Then for any $m \in \mathbb{N}$ we have

$$B \setminus B_m = \varphi(A) \setminus B_m \subseteq \varphi(A) \setminus \varphi(A_m) \subseteq \varphi(A \setminus A_m).$$

For any $m \in \mathbb{N}$ a set $\varphi^{-1}(m)$ is finite, so $\varphi(A \setminus A_m)$ is finite. Then the set $B \setminus B_m$ is finite. Hence, y is a P-filter.

Finally, we show that $y \in \mathcal{F}$. It is enough to show that $\{k \in \mathbb{N} : k \geq m\} \in y$ for any $m \in \mathbb{N}$.

Take any $m \in \mathbb{N}$ and put $m_1 = \max\limits_{i \leq m} \varphi^{-1}(i) + 1$. Since for any $k \in \mathbb{N}$ a set $\varphi^{-1}(k)$ is finite, $\varphi(\{k \in \mathbb{N} : k \geq m_1\}) \subseteq \{k \in \mathbb{N} : k \geq m\}$. Since $x \in \mathcal{F}$, we have that $\{k \in \mathbb{N} : k \geq m_1\} \in x$. Thus, $\{k \in \mathbb{N} : k \geq m\} \in y$. So, $y \in \mathcal{F}$. $\square$

Lemma 2. *Let $n \geq 2$, $x_1, \dots, x_n$ be P-filters from $\mathcal{F}$, $y = \Big\{ \bigcap\limits_{i=1}^{n} B_i \neq \varnothing : B_1 \in y_1, \dots, B_n \in y_n \Big\}$. Then y is a P-filter from $\mathcal{F}$.*

Proof. First we show that y is a filter.

1) Let $B_1, B_2 \in y$. We show that $B_1 \cap B_2 \in y$. According to the construction of y, for every $1 \leq i \leq n$ there exist sets $A_{1i}, A_{2i} \in x_i$ such that $B_1 = \bigcap\limits_{i=1}^{n} A_{1i}, B_2 = \bigcap\limits_{i=1}^{n} A_{2i}$. Then

$$B_1 \cap B_2 = \Big(\bigcap_{i=1}^{n} A_{1i} \Big) \bigcap \Big(\bigcap_{i=1}^{n} A_{2i} \Big) = \bigcap_{i=1}^{n} (A_{1i} \cap A_{2i}).$$

Since $A_{1i} \cap A_{2i} \in x_i$ for every $1 \leq i \leq n$, we get $B_1 \cap B_2 \in y$.

2) Let $B_1 \in y$ and $B_1 \subseteq B$. Show that $B \in y$. Since $B_1 \in y$, for every $1 \leq i \leq n$ there exists $A_i \in x_i$ such that $B_1 = \bigcap\limits_{i=1}^{n} A_i$. For every $1 \leq i \leq n$ we have that $A_i \subseteq A_i \cup B$ and therefore $A_i \cup B \in x_i$. Then

$$B = B \cup \Big(\bigcap_{i=1}^{n} A_i \Big) = \bigcap_{i=1}^{n} (A_i \cup B).$$

Hence, $B \in y$.

So, y is a filter. Now we show that y is a P-filter. Let $(B_m)_{m=1}^{\infty}$ be a sequence of sets from y. Then for any $m \in \mathbb{N}$ and for every $1 \leq i \leq n$ there exists a set $A_{mi} \in x_i$ such that $B_m = \bigcap\limits_{i=1}^{n} A_{mi}$. Since x_i is a P-filter for every $1 \leq i \leq n$, there exists a set $A_i \in x_i$ such that for any $m \in \mathbb{N}$ the set $A_i \setminus A_{mi}$ is finite. Put $B = \bigcap\limits_{i=1}^{n} A_i$. It is clear that $B \in y$. Then for any $m \in \mathbb{N}$ we have

$$B \setminus B_m = \Big(\bigcap_{i=1}^{n} A_i \Big) \setminus \Big(\bigcap_{i=1}^{n} A_{mi} \Big) = \bigcap_{i=1}^{n} (A_i \setminus A_{mi}).$$

For any $m \in \mathbb{N}$ and for every $1 \leq i \leq n$ the set $A_i \setminus A_{mi}$ is finite and so the set $B \setminus B_m$ is finite. Hence, y is a P-filter.

Finally, we show that $y \in \mathcal{F}$. It is enough to show that $\{k \in \mathbb{N} : k \geq m\} \in y$ for any $m \in \mathbb{N}$. Since y contains the intersections of sets from $x_1, \dots, x_n$ and $\{k \in \mathbb{N} : k \geq m\} \in x_i$ for any $m \in \mathbb{N}$ and for each $1 \leq i \leq n$, we obtain $\{k \in \mathbb{N} : k \geq m\} \in y$. So, $y \in \mathcal{F}$. □

Proposition 1. *Let $n \geq 2$. The following statements are equivalent.*

(i) *Any two P-filters from $\mathcal{F}$ are near coherent.*

(ii) *For every $n \geq 2$ any n P-filters from $\mathcal{F}$ are near coherent.*

(iii) *For some $n \geq 2$ any n P-filters from $\mathcal{F}$ are near coherent.*

Proof. $(i) \Rightarrow (ii)$ Assume that any two P-filters from $\mathcal{F}$ are near coherent. Show that any n P-filters from $\mathcal{F}$ are near coherent. To prove this we use the mathematical induction method.

It is obvious that for $n=2$ the statement is true.

Let this be true for some $n\in\mathbb{N}$. Show that the statement is valid for arbitrary P-filters $x_1,\dots,x_n,x_{n+1}$ from $\mathcal{F}$. According to the induction assumption, $x_1,\dots,x_n$ are near coherent, so there exists a function $\varphi:\mathbb{N}\to\mathbb{N}$ such that for any $m\in\mathbb{N}$ the set $\varphi^{-1}(m)$ is finite and $\bigcap\limits_{i=1}^{n}\varphi(A_i)\neq\varnothing$ for arbitrary sets $A_1\in x_1,\dots,A_n\in x_n$.

For any $i\le n+1$ we set $y_i=\{B\subseteq\mathbb{N}:\ \text{there exists a set } A\in x_i \text{ such that } \varphi(A)\subseteq B\}$. According to Lemma 1, y_i is a P-filter from $\mathcal{F}$. According to the choice of the function φ, we have $\bigcap\limits_{i=1}^{n}B_k\neq\varnothing$ for any $B_1\in y_1,\dots,B_n\in y_n$.

Set $y=\left\{\bigcap\limits_{i=1}^{n}B_i : B_1\in y_1,\dots,B_n\in y_n\right\}$. By Lemma 2 we have that y is a P-filter. According to item (i), y_{n+1} and y are near coherent. So there exists a function $\psi:\mathbb{N}\to\mathbb{N}$ such that for any $m\in\mathbb{N}$ the set $\psi^{-1}(m)$ is finite and $\psi(A)\cap\psi(B)\neq\varnothing$ for arbitrary sets $A\in y_{n+1}, B\in y$.

Set $h=\psi\circ\varphi$. Let $A_1\in x_1,\dots,A_n\in x_n, A_{n+1}\in x_{n+1}$ be arbitrary sets. Since $\bigcap\limits_{i=1}^{n}\varphi(A_i)\in y$ and $\varphi\left(A_{n+1}\right)\in y_{n+1}$, we have

$$\bigcap_{i=1}^{n+1}h(A_i)=\bigcap_{i=1}^{n+1}\psi(\varphi(A_i))\supseteq\psi\Big(\bigcap_{i=1}^{n}\varphi(A_i)\Big)\bigcap\psi(\varphi(A_{n+1}))\neq\varnothing.$$

Therefore, $x_1,\dots,x_n,x_{n+1}$ are near coherent. Thus, according to the mathematical induction, any n P-filters from $\mathcal{F}$ are near coherent.

$(ii)\Rightarrow(iii)$ It is obvious.

$(iii)\Rightarrow(i)$ Assume that for some $n\ge2$ any n P-filters from $\mathcal{F}$ are near coherent. Let x,y be P-filters from $\mathcal{F}$. We consider the following n P-filters $x,y,y,\dots,y$ from $\mathcal{F}$. According to the assumption, these n P-filters are near coherent. Then x,y are near coherent. □

Lemma 3. *Let $n\ge2$, $m\in\mathbb{N}$, $x_1,\dots,x_n$ be filters from $\mathcal{F}$ that are not near coherent. Then there exist sets $A_1\in x_1,\dots,A_n\in x_n$ such that for any point $(a_1,\dots,a_n)\in\prod_{k=1}^{n}A_k$ there exist numbers $i,j\in\{1,\dots,n\}$ such that $|a_i-a_j|>m$.*

Proof. For any $1\le k\le m+1$ we consider the function $\varphi_k:\mathbb{N}\to\mathbb{N}$ defined by

$$\varphi_k(y)=\left[\frac{y+k+m-1}{m+1}\right],$$

where $[x]$ means the integer part of $x\in\mathbb{R}$.

Since the filters $x_1,\dots,x_n$ are not near coherent, for any $1\le k\le m+1$ there exist sets $A_{k1}\in x_1,\dots,A_{kn}\in x_n$ such that $\bigcap\limits_{i=1}^{n}\varphi_k\left(A_{ki}\right)=\varnothing$.

For any $i\le n$, put $A_i=\bigcap\limits_{k=1}^{m+1}A_{ki}$. Let us note that $A_i\in x_i$ and $\bigcap\limits_{i=1}^{n}\varphi_k(A_i)=\varnothing$ for any $1\le k\le m+1$. For any $j\le n$, take $a_j\in A_j$ and $i\le n$ such that $a_i=\min\{a_j:1\le j\le n\}$. Now chose $1\le k\le m+1$ such that the number $a_i+k+m-1$ is divisible by $m+1$, i.e. $\frac{a_i+k+m+1}{m+1}\in\mathbb{N}$. Then for $y\in\mathbb{N}\cap[a_i,a_i+m]$ we have $\varphi_k(y)=\varphi_k(a_i)$.

Since $\bigcap\limits_{j=1}^{n}\varphi_k(A_j)=\varnothing$ and $(a_1,\dots,a_n)\in\prod\limits_{j=1}^{n}A_j$, there exists $j\le n$ such that $\varphi_k(a_j)\neq\varphi_k(a_i)$. Then $a_j\notin[a_i,a_i+m]$. According to the choice of a_j, we have $a_i\le a_j$. Hence, $a_j>a_i+m$. □

2 Necessity

Let $n \geq 2, X_1, \ldots, X_n$ be topological spaces, $X = \prod\limits_{i=1}^{n} X_i$, $\mathcal{S}_n = \{S \subset \{1, \ldots, n\} : S \neq \varnothing\}$, $S \in \mathcal{S}_n$ and $T = \{1, \ldots, n\} \setminus S$, $Y = \prod\limits_{s \in S} X_s$, $Z = \prod\limits_{t \in T} X_t$. For every point $x = (x_1, \ldots, x_n) \in X$, let us denote $x_{|_S} = (x_s)_{s \in S}$ and $x_{|_T} = (x_t)_{t \in T}$. It is obvious that $x_{|_S} \in Y$, $x_{|_T} \in Z$, and the function $\varphi_S : X \to Y \times Z$, defined by

$$\varphi_S(x) = \left(x_{|_S}, x_{|_T}\right),$$

is a homeomorphism.

A function $f : X \to \mathbb{R}$ is called *S-continuous at the point $x_0 \in X$* if the function $f_S : Y \times Z \to \mathbb{R}$, defined by

$$f_S(y, z) = f\left(\varphi_S^{-1}(y, z)\right),$$

is continuous with respect to the variable y at the point $\varphi_S(x_0)$. If a function f is S-continuous at every point $x \in X$, then f is called *S-continuous*.

Let $\mathcal{S} \subseteq \mathcal{S}_n$. A function $f : X \to \mathbb{R}$ is called *separately continuous by groups from the system $\mathcal{S}$* or separately *$\mathcal{S}$-continuous* if f is S-continuous for every $S \in \mathcal{S}$.

A function $f : X \to \mathbb{R}$ is called *strongly separately continuous* if it is separately $\mathcal{S}_n$-continuous.

It is obvious that every continuous function at a point is separately $\mathcal{S}$-continuous at this point. Moreover, a function $f : X \to \mathbb{R}$ is $\{k\}$-continuous for every $k \in \{1, \ldots, n\}$ if and only if f is separately continuous.

For $m < n$, a set $E \subseteq X$ is called *m separately finite* if for an arbitrary set $S \in \mathcal{S}_n$ such that the cardinality of S, denoted $|S|$, is equal m, and for any point $z \in \prod_{t \in T} X_t$, where $T = \{1, \ldots, n\} \setminus S$, the set $\left\{y \in \prod_{s \in S} X_s : (y, z) \in E\right\}$ is finite.

A set $E \subseteq X$ is called *strongly separately finite* if E is $(n-1)$ separately finite.

The following property is obvious.

Proposition 2. *Let $E \subseteq A = \prod_{i=1}^{n} A_i \subseteq X$ and $m < n$. Then the set E is m separately finite if for any $S \in \mathcal{S}_n$ with $|S| = m$, and for any point $z \in \prod_{t \in T} A_t$, where $T = \{1, \ldots, n\} \setminus S$, the set $\left\{y \in \prod_{s \in S} A_s : (y, z) \in E\right\}$ is finite.*

Let us look for the propositions of strongly separately finite sets.

Lemma 4. *Let $n \geq 2$ and $E \subseteq \mathbb{N}^n$ be strongly separately finite set. Then there exists a sequence $(C_k)_{k=1}^{\infty}$ of pairwise disjoint finite subsets of $\mathbb{N}$ such that*

$$\mathbb{N} = \bigsqcup_{k=1}^{\infty} C_k \quad \text{and} \quad E \subseteq \bigcup_{k=1}^{\infty} \bigcup_{k \leq k_1 \leq \cdots \leq k_n \leq k+1} \left(C_{k_1} \times \cdots \times C_{k_n}\right) = \bigcup_{|k_i - k_j| \leq 1} \left(C_{k_1} \times \cdots \times C_{k_n}\right).$$

Proof. For any two different $i, j \leq n$ and for every $m \in \mathbb{N}$, put

$$A(i, j, m) = \Big\{k \in \mathbb{N} : \text{there exists } \left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\right) \in \mathbb{N}^{n-2}$$
$$\text{such that } \left(x_1, \ldots, x_{i-1}, k, x_{i+1}, \ldots, x_{j-1}, m, x_{j+1}, \ldots, x_n\right) \in E\Big\}.$$

Notice that every set $A(i, j, m)$ is finite because E is strongly separately finite.

We construct the sequences $(D_k)_{k=0}^{\infty}$ and $(C_k)_{k=1}^{\infty}$ of sets as follows:

$$D_0 = \varnothing, \qquad D_{k+1} = D_k \bigcup \{k\} \bigcup \Bigg(\bigcup_{m \in D_k} \bigcup_{\substack{1 \le i,j \le n \\ i \ne j}} A(i,j,m) \Bigg),$$

$$C_{k+1} = D_{k+1} \setminus D_k$$

for every $k \in \mathbb{N}$.

Since $k \in D_{k+1}$ for any $k \in \mathbb{N}$ and $C_i \cap C_j = \varnothing$ for any $i \neq j$, we have $\bigsqcup\limits_{k=1}^{\infty} C_k = \mathbb{N}$.

Now, we show that

$$E \subseteq \bigcup_{|k_i - k_j| \le 1} (C_{k_1} \times \cdots \times C_{k_n}).$$

Fix any point $(x_1, \ldots, x_n) \in E$ and put $k = \min\{m \in \mathbb{N} : \{x_1, \ldots, x_n\} \cap D_m \neq \varnothing\}$. Without loss of the generality we can assume that $x_1 \in D_k$. Then $x_1 \in C_k$ and for any $2 \le i \le n$ we have that $x_i \in A(k, 1, x_1)$. Thus, $x_i \in D_{k+1}$ for any $2 \le i \le n$. Taking into account the choice of k we obtain $x_i \in D_{k+1} \setminus D_{k-1}$ for any $2 \le i \le n$. So, $x_1 \in C_1$ and $x_i \in D_{k+1} \setminus D_{k-1} = C_k \cup C_{k+1}$ for every $2 \le i \le n$. Thus, $(x_1, \ldots, x_n) \in \bigcup\limits_{|k_i - k_j| \le 1} (C_{k_1} \times \cdots \times C_{k_n})$. □

Theorem 1. *Let $n \ge 2$ and $u_1, \ldots, u_n$ be P-filters from $\mathcal{F}$, $X_i = \mathbb{N}_{u_i}$ for every $i \le n$, $u = (u_1, \ldots, u_n)$, $X = \prod_{i=1}^n X_i$. Let $f : X \to \mathbb{R}$ be a strongly separately continuous function such that $D(f) = \{u\}$. Then there exists strongly separately finite set $E \subseteq X$ such that the characteristic function $\chi|_E : X \to \{0,1\}$ is discontinuous at the point u.*

Proof. The function f is discontinuous at u, so there exists $\varepsilon > 0$ such that $\omega_f(u) = \varepsilon > 0$. Since f is strongly separately continuous, for any set $S \in \mathcal{S}_n$ the function $f_S : X_S \to \mathbb{R}$, $f_S(y) = f(y, u|_T)$, where $T = \{1, \ldots, n\} \setminus S$ and $X_S = \prod_{i \in S} X_i$, is continuous at $u|_S$. Therefore, for any $i \in S$ there exists $A(i, S) \in u_i$ such that for any point $y \in \prod_{i \in S} A(i, S)$ we have $|f_S(y) - f_S(u|_S)| < \frac{\varepsilon}{6}$.

For any $i \in \{1, \ldots, n\}$, denote $S_n(i) = \{S \in \mathcal{S}_n : i \in S\}$ and $A_i = \bigcap\limits_{S \in \mathcal{S}_n(i)} A(i, S)$. Notice that $A_i \in u_i$ as the intersection of finite number of sets from u_i. Moreover, $A = \prod_{i=1}^n A_i$. Then for any $S \in \mathcal{S}_n$ and for every $y \in \prod_{i \in S} A_i$ we have that $|f_S(y) - f_S(u|_S)| < \frac{\varepsilon}{6}$, i.e.

$$|f(y, u|_T) - f(u)| < \frac{\varepsilon}{6}. \tag{1}$$

Analogously, since f is strongly separately continuous, for any set $S \in \mathcal{S}_n$ and for any point $x \in \prod_{i \in T} A_i$ the function $f_{S,x} : X_S \to \mathbb{R}, f_{S,x}(y) = f(y, x)$, where $T = T_S = \{1, \ldots, n\} \setminus S$, is continuous at the point $u|_S$. So for any $i \in S$ there exists a set $A(i, S, x) \subseteq A_i$, $A(i, S, x) \in u_i$ such that for any point $y \in \prod_{i \in S} A(i, S, x)$ we have that $\left| f_{S,x}(y) - f_{S,x}(u|_S) \right| < \frac{\varepsilon}{6}$, i.e.

$$\left| f(y, x) - f\left(u|_S, x\right) \right| < \frac{\varepsilon}{6}. \tag{2}$$

For any $i \in \{1, \ldots, n\}$ the set $\mathcal{L}_i = \{(S, x) : S \in \mathcal{S}_n(i), x \in \prod_{j \in T_S} A_j\}$ is countable and so the family of sets $(A(i, S, x) : (S, x) \in \mathcal{L}_i)$ is countable for every $i \in \{1, \ldots, n\}$. Since $A(i, S, x) \in u_i$ and u_i is a P-filter for every $i \in \{1, \ldots, n\}$, there exists a set $B_i \subseteq A_i$ such that $B_i \in u_i$ and the set $B_i \setminus A(i, S, x)$ is finite for any $(S, x) \in \mathcal{L}_i$.

Put $B=\prod_{i=1}^{n} B_i$ and $E=\left\{x\in B : |f(u)-f(x)|>\frac{\varepsilon}{3}\right\}$. Let us prove that the set E is strongly separately finite. Using the induction with respect to m, we show that for every $1\le m\le n-1$ the set E is m separately finite.

First show that the set E is separately finite. Fix $i\in\{1,\dots,n\}$, put $S=\{i\}$, $T=T_S$, and fix $z\in\prod_{j\in T} A_j$. It is enough to show that the set $E(S,z)=\{y\in A_i : (y,z)\in E\}$ is finite. To do this, let us show that $E(S,z)\subseteq B_i\setminus A(i,S,z)$.

Since $E\subseteq B$, $E(S,z)\subseteq B_i$. Assume that there exists $y\in E(S,z)\cap A(i,S,z)$. Then $(y,z)\in E$. Therefore, $\left|f(u)-f(y,z)\right|>\frac{\varepsilon}{3}$.

On the other hand, according to (1) and (2), we have

$$\left|f(u)-f(y,z)\right|\le\left|f(u)-f(u_i,z)\right|+\left|f(u_i,z)-f(y,z)\right|<\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3}.$$

We got a contradiction and so $E(S,z)\cap A(i,S,z)=\varnothing$. Therefore $E(S,z)\subseteq B_i\setminus A(i,S,z)$ and the set $E(S,z)$ is finite. Thus, the set E is separately finite.

Let $1\le m<n-1$ and the set E is m separately finite, so for any set $S\in\mathcal{S}_n$ such that $|S|=m$ and for any point $z\in\prod_{j\in T_S} A_j$ the set $E(S,z)=\left\{y\in\prod_{i\in S} A_i : (y,z)\in E\right\}$ is finite. We show that E is $(m+1)$ separately finite.

Fix an arbitrary $S\in\mathcal{S}$, $|S|=m+1$, $T=T_S$ and $z\in\prod_{i\in T} A_i$. Show that the set $E(S,z)$ is finite. For any point $y\in\prod_{i\in S} A(i,S,z)\subseteq\prod_{i\in S} A_i$ according to (1) and (2), we have

$$\left|f(u)-f(y,z)\right|\le\left|f(u)-f(u|_S,z)\right|+\left|f(u|_S,z)-f(y,z)\right|<\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3},$$

i.e. $(y,z)\notin E$ and $(y,z)\notin E(S,z)$. Since $E\subseteq B=\prod_{i=1}^{n} B_i$, $E(S,z)\subseteq\prod_{i\in S} B_i$. Thus,

$$E(S,z)\subseteq\Big(\prod_{i\in S} B_i\Big)\setminus\Big(\prod_{i\in S} A(i,S,z)\Big)=\bigcup_{i\in S}\Big\{y\in\prod_{j\in S\setminus\{i\}} B_j\times(B_i\setminus A(i,S,z)) : (y,z)\in E\Big\}.$$

For every $i\in S$ put $S_i=S\setminus\{i\}$ and

$$F_i=\Big\{y\in\prod_{j\in S_i} B_j\times(B_i\setminus A(i,S,z)) : (y,z)\in E\Big\}.$$

According to what we proved above, we have that $E(S,z)\subseteq\bigcup\limits_{i\in S} F_i$.

It remains to show that all sets F_i are finite. Fix $i\in S$. The set $B_i\setminus A(i,S,z)$ is finite. For every $k\in B_i\setminus A(i,S,z)$, according to the induction assumption, the set

$$E\big(S_i,(k,z)\big)=\Big\{y\in\prod_{j\in S_j} B_j : (y,k,z)\in E\Big\}$$

is finite. Since $F_i=\bigcup\limits_{k\in B_i\setminus A(i,S,z)} E\left(S_i,(k,z)\right)$, the set F_i is finite. Thus, the set $E(S,z)$ is finite.

So, the set E is $(m+1)$ separately finite. According to the mathematical induction, for any $1\le m<n$ the set E is m separately finite. Therefore, the set E is strongly separately finite.

Not we show that $\chi|_E$ is discontinuous at the point u. For every $i\le n$ take an arbitrary neighborhood V_i of u_i. Since $B_i\cup\{u_i\}$ is a neighborhood of the point u_i, $U_i=V_i\cap(B_i\cup\{u_i\})$ is a neighborhood of u_i. Put $U=\prod_{i=1}^{n} U_i$.

It follows from the inequality $\sup\limits_{x\in U}\left\{\left|f(u)-f(x)\right|\right\}\ge\frac{\omega_f(U)}{2}=\frac{\varepsilon}{2}>\frac{\varepsilon}{3}$ that there exists a point $y\in U$ such that $\left|f(u)-f(y)\right|>\frac{\varepsilon}{3}$. We show that $y=(y_1,\dots,y_n)\in B=\prod_{i=1}^{n} B_i$, i.e. $y_i\ne u_i$ for every $i\le n$.

Assume that $S = \{i \neq n : y_i = u_i\} \neq \varnothing$. It is clear that in this case $T = T_S \neq \varnothing$. Then $y|_T \in \prod\limits_{i \in T} B_i$ and according to (1), we get $|f(u) - f(y)| = |f(u) - f(y|_T, u|_S) < \frac{\varepsilon}{6}$. We obtained a contradiction.

Hence, $y \in B$ and $y \in E$. It means that $E \cap U \neq \varnothing$. Taking into account the arbitrariness of V_i, we got that $u \in \overline{E}$ and this means that $\chi|_E$ is discontinuous at the point u. □

Theorem 2. *Let $n \geq 2$, $x_1, \dots, x_n$ be P-filters from $\mathcal{F}$ that are not near coherent, $X_i = \mathbb{N}_{u_i}$ for every $i \leq n$. Then every strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ is continuous.*

Proof. Assume that there exists a strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ such that f is discontinuous. Since the function f is strongly separately continuous and all points except $(x_1, \dots, x_n)$ are isolated at the appropriate spaces, $D(f) = \{x_1, \dots, x_n\}$.

According to Theorem 1, there exists a strongly separately finite set $E \subseteq \mathbb{N}^n$ such that the function $\chi|_E$ is discontinuous at $(x_1, \dots, x_n)$.

Using Lemma 4, we take a sequence $(C_k)_{k=1}^{\infty}$ of pairwise disjoint finite subsets of $\mathbb{N}$ such that

$$\mathbb{N} = \bigsqcup_{k=1}^{\infty} C_k \quad \text{and} \quad E \subseteq \bigcup_{|k_i - k_j| \leq 1} (C_{k_1} \times \dots \times C_{k_n}).$$

Consider the function $\varphi : \mathbb{N} \to \mathbb{N}$ defined by $\varphi(m) = k$ for any $k \in \mathbb{N}$ and $m \in C_k$. According to [12, Theorem 5.3.2], $x'_1 = \varphi(x_1), \dots, x'_n = \varphi(x_n)$ are P-filters from $\mathcal{F}$. Since $x_1, \dots, x_n$ are not near coherent, $x'_1, \dots, x'_n$ are not near coherent. Therefore, by Lemma 3 there exist $A'_1 \in x'_1, \dots, A'_n \in x'_n$ such that for any point $(a_1, \dots, a_n) \in \prod_{i=1}^n A'_i$ there exist numbers $i, j \leq n$ such that $|a_i - a_j| > 1$.

Put $A = \varphi^{-1}(A'_1), \dots, A_n = \varphi^{-1}(A'_n)$. For any point $(y_1, \dots, y_n) \in \prod_{i=1}^n A_i$ we have that $(y_1, \dots, y_n) \notin \bigcup\limits_{|k_i - k_j| \leq 1} (C_{k_1} \times \dots \times C_{k_n})$.

Thus $\prod_{i=1}^n A_i \cap E = \varnothing$ and this contradicts that the function $\chi|_E$ is discontinuous at the point $(x_1, \dots, x_n)$. So, our assumption is wrong and the function f is continuous. □

3 Sufficiency

Proposition 3. *Let $n \geq 2$, $X_1, X_2, \dots, X_n$ be topological spaces, for every $i \leq n$ a point x_{i0} be a non-isolated G_δ-point in the space X_i and $\varphi_i : X_i \to [0,1]$ be continuous function such that $\{x_{i0}\} = \varphi_i^{-1}(0)$ and the cardinality of a set $\{i \leq n :$ for any neighborhood U_i of the point x_{i0} a set $\varphi_i(U_i)$ is a neighborhood of 0 at $[0,1])\}$ is bigger or equal $n-1$. Then there exists a strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ such that $D(f) = \{(x_{10}, x_{20}, \dots, x_{n0})\}$.*

Proof. Without loss of the generality we can assume that for every $i < n$ and for any neighborhood U_i of x_{i0} the set $\varphi_i(U_i)$ is a neighborhood of 0 at $[0,1]$.

Consider the function $f : \prod_{i=1}^n X_i \to \mathbb{R}$, defined by

$$f(x_1, \dots, x_n) = \begin{cases} \frac{n\varphi_1(x_1)\dots\varphi_n(x_n)}{\varphi_1^n(x_1) + \dots + \varphi_n^n(x_n)}, & (x_1, \dots, x_n) \neq (x_{10}, \dots, x_{n0}), \\ 0, & (x_1, \dots, x_n) = (x_{10}, \dots, x_{n0}). \end{cases}$$

The function f is continuous at each point of the set $\prod_{i=1}^{n} X_i \setminus \{(x_{10},\dots,x_{n0})\}$ as a composition of continuous functions. Moreover, for any $i \le n$ the function

$$f_{x_{i0}} : \prod_{j=1, j\neq i}^{n} X_j \to \mathbb{R}, \qquad f_{x_{i0}}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) = f(x_1,\dots,x_{i-1},x_{i0},x_{i+1},\dots,x_n),$$

equals to 0 and it is continuous. So the function f is strongly separately continuous.

It remains to show that the function f is discontinuous at the point $(x_{10},\dots,x_{n0})$. Let $0 < \varepsilon < 1$. For any $i \le n$ take an arbitrary neighborhood U_i of the point x_{i0}. For every $i < n$ the set $\varphi_i(U_i)$ is a neighborhood of 0 at $[0,1]$. Therefore, there exists $\delta_i > 0$ such that $[0,\delta_i) \subseteq \varphi_i(U_i)$. Put $\delta = \min\limits_{1\le k<n} \delta_i$.

The function φ_n is continuous at the point x_{n0} and $\varphi_n(x_{n0}) = 0$. Thus there exists a neighborhood V_n of the point x_{n0} such that for any point $x_n \in V_n$ we have $0 \le \varphi_n(x_n) < \delta$. The point x_{n0} is non-isolated in X_n, so there exists a point $x'_n \in (U_n \cap V_n) \setminus \{x_{n0}\}$. Then $\varphi_n(x'_n) > 0$, because $\{x_{n0}\} = \varphi_n^{-1}(0)$.

According to the choice of δ, for any $i < n$ there exists a point $x'_i \in U_i$ such that $\varphi_i(x'_i) = \varphi_n(x'_n)$. Thus,

$$\left| f\,(x'_1,\dots,x'_n) - f(x_{10},\dots,x_{n0}) \right| = f(x'_1,\dots,x'_n) = 1 > \varepsilon$$

and $D(f) = \{(x_{10},\dots,x_{n0})\}$. □

Proposition 4. *Let $n \ge 2$ and $x_1, x_2, \dots, x_n$ be P-filters from $\mathcal{F}$ that are near coherent, $X_i = \mathbb{N}_{u_i}$ for every $i \le n$. Then there exists a set $H \subset \mathbb{N}^n$ such that the characteristic function $f : X = \prod_{i=1}^{n} X_i \to \mathbb{R}, f = \chi|_H$, is strongly separately continuous with $D(f) = \{(x_1,\dots,x_n)\}$.*

Proof. Since $x_1,\dots,x_n$ are near coherent, there exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi^{-1}(m)$ is finite for any $m \in \mathbb{N}$ and $\bigcap\limits_{i=1}^{n} \varphi(A_i) \neq \varnothing$ for every $i \le n$ and for any set $A_i \in x_i$.

Consider the set $H = \{(u_1,\dots,u_n) \in \mathbb{N}^n : \varphi(u_1) = \dots = \varphi(u_n)\}$ and the function $f : X \to \mathbb{R}$, $f = \chi|_H$. We show that f is strongly separately continuous. For every $i \le n$ the function f_{x_i} equals to 0 and so it is continuous. Thus the function f is continuous at all points from the set $\mathbb{N}^n$, because all points from the set $\mathbb{N}^n$ are isolated.

Take any point $u_1 \in \mathbb{N}$ and show that the function f_{u_1} is continuous on $\prod_{j=2}^{n} X_i \setminus \mathbb{N}^{n-1}$. Let $(u_2,\dots,u_n) \in \prod_{j=2}^{n} X_i \setminus \mathbb{N}^{n-1}$, i.e. there exists a number $2 \le i \le n$ such that $u_i = x_i$. According to the choice of the function φ, we have that $\varphi^{-1}(\varphi(u_1))$ is finite. This means that there exists a number m_1 such that $\varphi(m) \neq \varphi(u_1)$ for every $m > m_1$.

Since $x_i \in \mathcal{F}$, $U_i = \{m \in \mathbb{N} : m > m_1\} \in x_i$, so $U_i \cup \{x_i\}$ is a neighborhood of x_i. Moreover, $\varphi(U_i) \cap \{\varphi(u_1)\} = \varnothing$.

Let U_j be an arbitrary neighborhood of the point u_j for every $2 \le j \le n, j \neq i$. According to the choice of U_i, $f_{u_1} = 0$ on the open set $\prod_{j=2}^{n} U_j$ and so the function f_{u_1} is continuous at the point $(u_2,\dots,u_n)$. Taking into account the arbitrariness of the point $(u_2,\dots,u_n)$, the function f_{u_1} is continuous on $\prod_{i=2}^{n} X_i \setminus \mathbb{N}^{n-1}$. In the same way we can get that the function f_{u_i} is continuous for any $2 \le i \le n$. Thus the function f is strongly separately continuous.

Now we show that f is discontinuous at $(x_1,\dots,x_n)$. We take $U_i \in x_i$ is an arbitrary set for every $i \le n$. According to the choice of φ, there exists the point $(u_1,\dots,u_n) \in \prod_{i=1}^{n} U_i$ such that $\varphi(u_1) = \dots = \varphi(u_n)$. Then $f(u_1,\dots,u_n) = 1$, so $\omega_f(x_1,\dots,x_n) \ge 1$. Hence, $D(f) = \{(x_1,\dots,x_n)\}$. □

Proposition 5. *Let $n \geq 2$ and $x_1, \dots, x_n$ be filters from $\mathcal{F}$ such that at least one is not a P-filter, $X_i = \mathbb{N}_{u_i}$ for every $i \leq n$, $X = \prod_{i=1}^n X_i$. Then there exists a set $H \subseteq \mathbb{N}^n$ such that the characteristic function $f : X \to \mathbb{R}, f = \chi|_H$, is strongly separately continuous with $D(f) = \{(x_1, \dots, x_n)\}$.*

Proof. Without loss of the generality we can assume that x_n is not a P-filter, i.e. there exists a sequence $(A_k)_{k=1}^\infty$ of sets $A_k \in x_n$ such that for any set $A \in x_n$ there exists an integer m such that $A \setminus A_m$ is infinite and this sequence is monotonically decreasing.

Put $H = \Big\{(u_1, \dots, u_n) \in \mathbb{N}^n : u_n \in \Big(A_1 \setminus A_{\min\limits_{1\leq i<n} u_i}\Big) \cap \Big[\max\limits_{1\leq i<n} u_i, \infty\Big)\Big\} \subset \mathbb{N}^n$ and $f = \chi|_H$. We show that f is desired. Fix $u_n \in \mathbb{N}$. Notice that

$$\Big\{(u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1} : (u_1, \dots, u_n) \in H\Big\} \subseteq \Big\{(u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1} : \max_{1\leq i<n} u_i \leq u_n\Big\}.$$

Therefore, this set is finite and so it is clopen in $\prod_{i=1}^{n-1} X_i$. Thus, the function f_{u_n} is continuous as the characteristic function of clopen set. Moreover, the function f is continuous at every point of a set $X_1 \times \cdots \times X_{n-1} \times \{u_n\}$, because the point $\{u_n\}$ is isolated at X_n.

Now, let $u_n = x_n$ and $(u_1, \dots, u_n) \in X \setminus \{(x_1, \dots, x_n)\}$. Then there exists an integer $i < n$ such that $u_i \in \mathbb{N}$. Without loss of the generality we can assume that $u_1 \in \mathbb{N}$ and $u_1 = \min\{u_i : u_i \in \mathbb{N}, 1 \leq i < n\}$. Then

$$\{u_1\} \times X_2 \times \cdots \times X_{n-1} \times (A_{u_1} \cup \{x_n\}) \subseteq f^{-1}(0).$$

The set $\{u_1\} \times X_2 \times \cdots \times X_{n-1} \times (A_{u_1} \cup \{x_n\})$ is an open neighborhood of the point $(u_1, u_2, \dots, u_n)$, so the function f is continuous at the point $(u_1, \dots, u_n)$.

Thus, the function f is continuous at all points of the set $X \setminus \{(x_1, \dots, x_n)\}$. In particular, for any $i \leq n$ the function $f_{x_i} = 0$ is continuous. Thus, the function f is strongly separately continuous.

Now, we show that $D(f) = \{(x_1, \dots, x_n)\}$. Assume that the function f is continuous. Since $f(x_1, \dots, x_n) = 0$, for every $i \leq n$ there exists a set $B_i \in x_i$ such that $f(u_1, \dots, u_n) = 0$ for any point $(u_1, \dots, u_n) \in \prod_{i=1}^n B_i$.

Put $A = B_n \cap A_1$ and fix $m \in \mathbb{N}$. Since $\{k \in \mathbb{N} : k \geq m\} \in x_i$ for every $i < n$, there exists $m_i \geq m$ such that $m_i \in B_i$ for every $i < n$. Put $p = \min\limits_{1\leq i<n} m_i, q = \max\limits_{1\leq i<n} m_i$. It is obvious that $p > m, q > m$. Notice that $f(m_1, \dots, m_{n-1}, u_n) = 0$, i.e. $(m_1, \dots, m_{n-1}, u_n) \notin H$ for any $u_n \in A$. Moreover, taking into account the choice of p and q, we obtain

$$A \cap (A_1 \setminus A_p) \cap [q, \infty) = (A \setminus A_p) \cap [q, \infty] = \varnothing.$$

Thus, the set $A \setminus A_p$ is finite. Since $A_p \subseteq A_m$, the set $A \setminus A_m$ is finite. Since we took an arbitrary number $m \in \mathbb{N}$, we got a contradiction with condition that x_n is not a P-filter. Therefore, the function f is not continuous at $(x_1, \dots, x_n)$ and $D(f) = \{(x_1, \dots, x_n)\}$. □

The next corollary follows from the the Propositions 4 and 5.

Corollary 1. *Let $n \geq 2$. Let $x_1, x_2, \dots, x_n$ be filters from $\mathcal{F}$ such that one of the following conditions is true:*

1) at least one of x_k is not a P-filter;

2) $x_1, x_2, \dots, x_n$ are P-filters and they are near coherent.

For every $1 \le k \le n$ put $X_k = \mathbb{N}_{x_k}$. Then there exists a set $H \subset \mathbb{N}^n$ such that the characteristic function $f : X = \prod_{k=1}^n X_k \to \mathbb{R}, f = \chi|_H$, is strongly separately continuous with $D(f) = \{(x_1, \dots, x_n)\}$.

Theorem 3. *Let $n \ge 2$ and $X_1, \dots, X_n$ be completely regular spaces, x_{i0} be a non-isolated G_δ-point in X_i for every $i \le n$ and any two P-filters from $\mathcal{F}$ are near coherent.Then there exists a strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ such that $D(f) = \{(x_{10}, \dots, x_{n0})\}$.*

Proof. According to [12, Lemma 5.3.7], for every $i \le n$ there exists a continuous function $\varphi_i : X_i \to [0,1]$ such that $\varphi_i^{-1}(0) = \{x_{i0}\}$.

If there does not exist or there exists exactly one number i such that there exists an open neighborhood U_i of the point x_{i0} such that the set $\varphi_i(U_i)$ is not a neighborhood of 0 in $[0,1]$, then according to the Proposition 3 the theorem is true.

Now, let there exist at least two such numbers. Without loss of the generality we can assume that there exists the number $j \le n-1$ such that for every $j \le i \le n$ there exists an open neighborhood U_{i0} of x_{i0} such that $\varphi_i(U_{i0})$ is not a neighborhood of 0 in $[0,1]$. Then according to [12, Lemma 5.3.9], for every $j \le i \le n$ there exists a monotonically decreasing sequence $(F_{im})_{m=1}^\infty$ of clopen sets F_{im} in the space U_{i0} such that $F_{i1} = U_{i0}$ and $\{x_{i0}\} = \bigcap\limits_{m=1}^{\infty} F_{im}$. Moreover, since x_{i0} is non-isolated point, we can assume that $(F_{im})_{m=1}^\infty$ is strictly monotonically decreasing sequence.

For every $j \le i \le n$ we consider the surjective function $\psi_i : U_{i0} \setminus \{x_{i0}\} \to \mathbb{N}$ defined by $\psi_i(F_{im} \setminus F_{i,m+1}) = \{m\}$. For every $j \le i \le n$, put

$$u_i = \big\{\psi_i(U \setminus \{x_{i0}\}) : U \text{ is a neighborhood of the point } x_{i0}\big\}.$$

It is clear that u_i is a filter from $\mathcal{F}$.

For every $j \le i \le n$, put $\psi_i(x_{i0}) = u_i$. We obtain the function $\psi_i : U_{i0} \to \mathbb{N}_{u_i}$ which is continuous at the point x_{i0}. Moreover, since every set $F_{im} \setminus F_{i,m+1}$ is clopen, the function ψ_i is continuous at all points from $U_{i0} \setminus \{x_{i0}\}$.

According to Proposition 1, any $n-j+1$ P-filters from $\mathcal{F}$ are near coherent. It follows from Corollary 1 that there exists a set $A \subset \mathbb{N}^{n-j+1}$ such that the function $s : \prod_{i=j}^n \mathbb{N}_{u_i} \to [0,1]$, $s = \chi|_A$, is strongly separately continuous and $D(s) = \{(u_j, \dots, u_n)\}$.

Consider the function $\varphi_0 : \prod_{i=j}^n U_{i0} \to [0,1]$, $\varphi_0(x_j, \dots, x_n) = s(\psi_j(x_j), \dots, \psi_n(x_n))$. Since s is strongly separately continuous, the function φ_0 is strongly separately continuous. Moreover, φ_0 is continuous at all points from the set $\left(\prod_{i=j}^n U_{i0}\right) \setminus \{(x_{j0}, \dots, x_{n0})\}$ as composition of continuous functions. For every $j \le i \le n$ and for any neighborhood U_i of x_{i0}, the set $\psi_i(U_i)$ is a neighborhood of the point u_i in the space $\mathbb{N}_{u_i}$. Thus,

$$\omega_{\varphi_0}\left(\prod_{i=j}^n U_i\right) = \omega_s\left(\prod_{i=j}^n \psi_i(U_i)\right) \ge \omega_s(u_j, \dots, u_n).$$

Hence, the function φ_0 is discontinuous at $(x_{j0}, \dots, x_{n0})$ and $D(\varphi_0) = \{(x_{j0}, \dots, x_{n0})\}$.

Next we consider the following two cases.

1. Let $j = 1$. Since the space $\prod_{i=1}^n X_i$ is a completely regular space, there exists a continuous function $\theta : \prod_{i=1}^n X_i \to [0,1]$ such that $\theta(x_{10}, \dots, x_{n0}) = 1$ and $\theta(x_1, \dots, x_n) = 0$ if $(x_1, \dots, x_n) \notin \prod_{i=1}^n U_{i0}$.

Now we put $f : \prod_{i=1}^n X_i \to \mathbb{R}$,

$$f(x_1,\dots,x_n) = \begin{cases} \theta(x_1,\dots,x_n)\varphi_0(x_1,\dots,x_n), & (x_1,\dots,x_n) \in \prod\limits_{i=1}^n U_{i0}, \\ 0, & (x_1,\dots,x_n) \notin \prod\limits_{i=1}^n U_{i0}. \end{cases}$$

According to [12, Proposition 5.3.10], $D(f) = D(\varphi_0) \cap \prod_{i=1}^n U_{i0}$. Therefore, the function f is strongly separately continuous with $D(f) = \{(x_{i0},\dots,x_{n0})\}$. Hence, the function f satisfies the condition of theorem.

2. Let $1 < j < n$. For any $m \in \mathbb{N}$, let us set $A_m = \left(A \cap [1,m]^{n-j+1}\right) \setminus \bigsqcup\limits_{k=1}^{m-1} A_k$. It is obvious that $A = \bigsqcup\limits_{m=1}^{\infty} A_m$ and $A_p \neq A_q$ for $p \neq q$. Now we define $\varphi : \prod_{i=j}^n U_{i0} \to [0,1]$ by

$$\varphi(x_i,\dots,x_n) = \begin{cases} \frac{1}{m}, & (\psi_j(x_j),\dots,\psi_n(x_n)) \in A_m, \\ 0, & \varphi_0(x_j,\dots,x_n) = 0. \end{cases}$$

Let us show that φ is continuous function. For any $(x_j,\dots,x_n) \in \left(\prod_{i=j}^n U_{i0}\right) \setminus \{(x_{j0},\dots,x_{n0})\}$ such that $\varphi_0(x_j,\dots,x_n) = 0$, the function φ_0 is continuous at $(x_j,\dots,x_n)$. So there exists a neighborhood of this point such that $\varphi_0 = 0$ on it. Therefore, φ is continuous at this point.

According to the construction of A_m, for any $(x_j,\dots,x_n) \in \left(\prod_{i=j}^n U_{i0}\right) \setminus \{(x_{j0},\dots,x_{n0})\}$ such that $\varphi_0(x_j,\dots,x_n) \neq 0$, there exists an integer m such that $(\psi_j(x_j),\dots,\psi_n(x_n)) \in A_m$ and so $\psi_i(x_i) \in [1,m]$ for every $j \le i \le n$ and $x_i \in F_{i\psi_i(x_i)}$. For every $j \le i \le n$ the set $F_{i\psi_i(x_i)}$ is an open neighborhood of x_i and for every point $(x'_j,\dots,x'_n) \in \prod_{i=j}^n F_{i\psi_i(x_i)}$ we have

$$\left|\varphi(x'_j,\dots,x'_n) - \varphi(x_j,\dots,x_n)\right| = \left|\frac{1}{m} - \frac{1}{m}\right| = 0.$$

Thus, the function φ is continuous at the point $(x_j,\dots,x_n)$.

Now we need to show that the function φ is continuous at the point $(x_{j0},\dots,x_{n0})$. For any $\varepsilon > 0$ there exists a number $m \in \mathbb{N}$ such that $\dfrac{1}{m} < \varepsilon$. For any point $(x'_j,\dots,x'_n) \in \prod_{i=j}^n F_{im}$ we have

$$\left|\varphi(x'_j,\dots,x'_n) - \varphi(x_{j0},\dots,x_{n0})\right| = \varphi(x'_j,\dots,x'_n) \le \frac{1}{m} < \varepsilon.$$

So, the function φ is continuous at the point $(x_{i0},\dots,x_{n0})$ and therefore it is continuous.

Now consider the function $g : \prod_{i=1}^{j-1} X_i \times \prod_{i=j}^n U_{i0} \to \mathbb{R}$, defined by

$$g(x_1,\dots,x_n) = \begin{cases} \frac{j\varphi_1(x_1)\dots\varphi_{j-1}(x_{j-1})\varphi(x_j,\dots,x_n)}{\varphi_1^j(x_1)+\dots+\varphi_{j-1}^j(x_{j-1})+\varphi^j(x_j,\dots,x_n)}, & \varphi_1(x_1)+\dots+\varphi(x_j,\dots,x_n) > 0, \\ 0, & \varphi_1(x_1)+\dots+\varphi(x_j,\dots,x_n) = 0. \end{cases}$$

We show that the function g is strongly separately continuous. For any $i < j$ fix any point $x_i \in X_i \setminus \{x_{i0}\}$. Consider the function

$$g_{x_i} : \prod_{\substack{i=1 \\ i\neq j}}^{j-1} X_i \times \prod_{i=j}^n U_{i0} \to \mathbb{R}, \qquad g_{x_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) = g(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n).$$

The function g_{x_i} is continuous as composition of continuous functions, because $\varphi_i(x_i) > 0$. Moreover, for any $i \le n$ the function $g_{x_{i0}} \equiv 0$. Therefore $g_{x_{i0}}$ is continuous.

Now for every $j \le i \le n$ we fix any point $x_i \in U_{i0} \setminus \{x_{i0}\}$. Consider sets

$$H = \Big\{(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \prod_{l=j, l\neq i}^{n} U_{l0} : (\psi_j(x_j), \dots, \psi_i(x_i), \dots \psi_n(x_n)) \in A)\Big\}$$

and

$$G = \Big\{(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \prod_{l=j, l\neq i}^{n} U_{l0} : (\psi_j(x_j), \dots, \psi_k(x_i), \dots \psi_n(x_n)) \notin A)\Big\}.$$

Since the characteristic function $\chi|_A$ is strongly separately continuous, the sets H and G are open and $H \cup G = \prod\limits_{l=j, l\neq i}^{n} U_{l0}$. For any point $(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in H$ we have that $\varphi_{x_i}(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \neq 0$.

Thus, the function g_{x_i} is continuous at every pint from the set H as composition of continuous function. At the same time for any point $(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in G$ we have that $\varphi_{x_i}(x_j, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = 0$. Therefore, $g_{x_i} \equiv 0$ and g_{x_i} is continuous.

Hence, the function g is strongly separately continuous. It remains to show that f is discontinuous at the point $(x_{10}, \dots, x_{n0})$.

Let $0 < \varepsilon < 1$. For any $i \le n$ we take any neighborhood V_i of the point x_{i0}. For every $i < j$ the set $\varphi_i(V_i)$ is a neighborhood of the 0 in $[0, 1]$. Thus, there exists a number $\delta_i > 0$ such that $[0, \delta_i) \subseteq \varphi_k(V_i)$. Put $\delta = \min\limits_{1 \le i < j} \delta_i$.

The function φ is continuous at the point $(x_{j0}, \dots, x_{n0})$ and $\varphi(x_{j0}, \dots, x_{n0}) = 0$. Therefore, for every $k \le i \le n$ there exists a neighborhood V_i of the point x_{i0} such that for any point $(x_j, \dots, x_n) \in \prod_{i=j}^{n} V_i$ we have that $0 \le \varphi(x_j, \dots, x_n) < \delta$. For every $j \le i \le n$ we get $\psi_i(V_i) \in u_i$. Since $D(s) = \{(u_j, \dots, u_n)\}$ and $\operatorname{supp} s \subset \mathbb{N}^{n-i+1}$, there exist points $x'_j \in V_j \setminus \{x_{j0}\}, \dots, x'_n \in V_n \setminus \{x_{n0}\}$ such that $(\psi_j(x'_j), \dots, \psi_n(x'_n)) \in A$ and $\varphi(x'_i, \dots, x'_n) > 0$. It follows from the choice of δ, that for any $i < j$ there exists a point $x'_i \in V_i$ such that $\varphi_k(x'_i) = \varphi(x'_j, \dots, x'_n)$. Then $|g(x'_1, \dots, x'_n) - g(x_{10}, \dots, x_{n0})| = g(x'_1, \dots, x'_n) = 1 > \varepsilon$. Therefore, $D(g) = \{(x_{10}, \dots, x_{n0})\}$.

Since $\prod_{i=1}^{n} X_i$ is completely regular space, there exists a continuous function $\theta : \prod_{i=1}^{n} X_i \to [0, 1]$ such that $\theta(x_{10}, \dots, x_{n0}) = 1$ and $\theta(x_1, \dots, x_n) = 0$, if $(x_1, \dots, x_n) \notin \prod_{i=1}^{j-1} X_i \times \prod_{i=j}^{n} U_{i0}$. Now we consider the function $f : \prod_{i=1}^{n} X_i \to \mathbb{R}$, defined by

$$f(x_1, \dots, x_n) = \begin{cases} \theta(x_1, \dots, x_n) g(x_1, \dots, x_n), & (x_1, \dots, x_n) \in \prod\limits_{i=1}^{j-1} X_i \times \prod\limits_{i=j}^{n} U_{i0}, \\ 0, & (x_1, \dots, x_n) \notin \prod\limits_{i=1}^{j-1} X_i \times \prod\limits_{i=j}^{n} U_{i0}. \end{cases}$$

According to [12, Proposition 5.3.10], $D(f) = D(g) \cap \prod_{i=1}^{j-1} X_i \times \prod_{i=j}^{n} U_{i0}$. Thus, f is a strongly continuous function with $D(f) = \{(x_{10}, \dots, x_{n0})\}$. $\square$

The next theorem is the main result of this article and it follows immediately from the previous theorems.

Theorem 4. *The following statements are equivalent:*

(i) *for any $n \geq 2$, completely regular spaces $X_1, \ldots, X_n$ and non-isolated G_δ-points in corresponding spaces $x_{10} \in X_1, \ldots, x_{n0} \in X_n$, there exists a strongly separately continuous function $f : \prod_{i=1}^n X_i \to \mathbb{R}$ with $D(f) = \{(x_{10}, \ldots, x_{n0})\}$;*

(ii) *any two P-filters from $\mathcal{F}$ are near coherent.*

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Козловський М. *Розривні сильно нарізно неперервні функції багатьох змінних та майже когерентність двох P-фільтрів* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 469–483.

Ми розглядаємо поняття майже когерентності n P-фільтрів і показуємо, що майже когерентність довільних n P-фільтрів еквівалентна майже когерентності довільних двох P-фільтрів. Для довільного фільтра u на $\mathbb{N}$ через $\mathbb{N}_u$ ми позначаємо простір $\mathbb{N} \cup \{u\}$, в якому всі точки з $\mathbb{N}$ ізольовані і множини $A \cup \{u\}$, $A \in u$, є околами u. У статті введено поняття сильно нарізно скінченних множин. Для $X = \mathbb{N}_{u_1} \times \cdots \times \mathbb{N}_{u_n}$ доведено, що існування сильно нарізно неперервної функції $f : X \to \mathbb{R}$ з одноточковою множиною розривів $\{(u_1, \ldots, u_n)\}$ означає, що існує нарізно скінченна множина $E \subseteq X$ така, що характеристична функція $\chi|_E$ розривна в $(u_1, \ldots, u_n)$. Використовуючи даний факт ми довели, що існування сильно нарізно скінченної функції $f : X_1 \times \cdots \times X_n \to \mathbb{R}$ на добутку цілком регулярних просторів X_k із одноточковою множиною розривів $\{(x_1, \ldots, x_n)\}$, де x_k неізольована G_δ-точка в X_k, еквівалентне до майже когерентності P-фільтрів.

Ключові слова і фрази: нарізно неперервна функція, сильно нарізно неперервна функція, P-фільтр, обернена задача, одноточковий розрив.