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Approximation of functions from Hölder class by biharmonic Poisson integrals

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The biharmonic equation in Cartesian coordinates is considered for the case of the upper half-plane. The solution of such a fourth-order partial differential equation for given boundary conditions is represented in the form of an integral of the product of the function and the delta-shaped kernel, which in this paper plays the role of an approximating aggregate. In the paper, we found an exact equality for the upper bound of the deviation of Hölder class functions from the considered biharmonic Poisson operator in the uniform metric.

Key words and phrases: biharmonic equation, biharmonic Poisson integral, Hölder class.

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Denote as usual $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. Consider the biharmonic equation

$$\nabla^2 \left(\nabla^2 U\right) = 0 \tag{1}$$

in the Cartesian coordinate system for the upper half-plane, i.e. for $y > 0$. Let the following boundary conditions

$$\lim_{y\to+0} U\left(x,y\right) = f\left(x\right) \tag{2}$$

and

$$\lim_{y\to+0} \frac{\partial U\left(x,y\right)}{\partial y} = 0 \tag{3}$$

hold.

The fourth-order partial differential equation (1) with the boundary conditions (2) and (3) is a boundary value problem. The solution to the boundary value problem under consideration can be represented as an integral

$$\mathcal{B}\left(f;x,y\right) := \frac{2y^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(x+t)}{(t^2+y^2)^2}\,dt. \tag{4}$$

The positive operator (4) is also a functional of the function f and is called a biharmonic Poisson integral in the upper half-plane. It should be noted that the integral kernel in the expression (4) for the biharmonic Poisson operator is a special case of a more general kernel

$$\frac{A+Bx}{(x^2+\lambda^2)^2},$$

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which has already been studied earlier in [13].

Let C be the space of continuous and summable on the real axis functions f with the usual norm $\|f\| = \max_{x\in\mathbb{R}} |f(x)|$. Also, let H^α, $0 < \alpha < 1$, denote the Hölder class of functions $f \in C$ satisfying the condition

$$\left|f(x_1) - f(x_2)\right| \le |x_1 - x_2|^\alpha, \qquad x_1, x_2 \in \mathbb{R}. \tag{5}$$

Following [16, p. 198], consider the classical problem of finding an exact equality for the quantity

$$\mathcal{E}\big(H^\alpha, \mathcal{B}(y)\big)_C = \sup_{f\in H^\alpha} \big\|f(x) - \mathcal{B}(f; x, y)\big\|_C \tag{6}$$

of upper bound on the deviation of functions of Hölder class H^α from the operator (4) in the space C.

The approximative properties of the biharmonic function (4), which belongs to the class of polyharmonic operators in the upper half-plane, are poorly studied. As for the polyharmonic operators in the unit disk, a considerable number of publications have been devoted to the study of the approximative properties of such approximating aggregates. In particular, the Abel-Poisson integrals [6, 18] and biharmonic Poisson integrals [3, 18] are well studied. In general, the problem of approximating functions by polyharmonic operators in the metric of the space S^p, $1 \le p < \infty$, was solved by M.F. Timan [17, p. 256].

Similar questions were also studied for the case of many variable functions. In the series of papers [8–12], approximative properties of the Taylor-Abel-Poisson summation method were studied. This method defines the operators $A_{\varrho,r}$ that possess the main properties of the Abel-Poisson and Taylor operators, however they can be also adapted to smoothness properties of functions of arbitrarily large order. In particular, in [8, 9], the authors proved the direct and inverse theorems on the approximation of functions of several variables by operators $A_{\varrho,r}$ in terms of K-functionals of functions generated by their radial derivatives in the integral metric.

The problem of the type (6) was frequently considered in the case when the role of the approximating aggregate is played by linear methods [1, 2, 14] of Fourier series summation that are given by infinite triangular numerical matrices (more detailed information on this issue is covered in the book [16]) and arbitrary sequences of functions [4, 7].

It is worth noting that most of the results concerning the approximative properties of biharmonic Poisson integrals for a unit disk are written in the form of asymptotic equalities [5, 19].

The purpose of this study is to find an exact equality for a quantity of the type (6) in the case when the biharmonic Poisson integral (4) for the upper half-plane acts as an approximating aggregate.

Theorem 1. *For any fixed $y > 0$ and $0 < \alpha < 1$, the exact equality*

$$\mathcal{E}\big(H^\alpha, \mathcal{B}(y)\big)_C = \frac{(1-\alpha)y^\alpha}{\cos\frac{\pi\alpha}{2}} \tag{7}$$

holds.

Proof. Using the operator (4), we will construct an integral representation

$$\mathcal{B}(f; x, y) - f(x) = \frac{2y^3}{\pi} \int_{-\infty}^{+\infty} \frac{f(x+t) - f(x)}{(t^2 + y^2)^2}\, dt. \tag{8}$$

With the help of formulas (8) and (5), we get the estimate

$$\mathcal{E}\big(H^{\alpha}, \mathcal{B}(y)\big)_C = \sup_{f \in H^{\alpha}} \big\| f(x) - \mathcal{B}(f; x, y) \big\|_C \leq \frac{4y^3}{\pi} \int_0^{+\infty} \frac{t^{\alpha}\, dt}{(t^2 + y^2)^2}. \tag{9}$$

In the class H^{α} there is a function $f(t) = |t|^{\alpha}$, which transforms the inequality (9) into an equality. So we have

$$\mathcal{E}\big(H^{\alpha}, \mathcal{B}(y)\big)_C = \frac{4y^3}{\pi} \int_0^{+\infty} \frac{t^{\alpha}\, dt}{(t^2 + y^2)^2}.$$

The last integral can be transformed using the substitution $t = yz$. Then we get

$$\mathcal{E}\big(H^{\alpha}, \mathcal{B}(y)\big)_C = \frac{4y^{\alpha}}{\pi} \int_0^{+\infty} \frac{z^{\alpha}\, dz}{(1 + z^2)^2}. \tag{10}$$

Let us evaluate the integral in formula (10). To do this, we will consider the integral

$$\int_0^{+\infty} \frac{z^a\, dz}{(1 + z^2)^2} = \int_0^{1} \frac{z^a\, dz}{(1 + z^2)^2} + \int_1^{+\infty} \frac{z^a\, dz}{(1 + z^2)^2}, \tag{11}$$

dependent on the parameter a.

Numerical values of the parameter a for which the integral (11) exists will be found below. We will transform the second integral on the right-hand side of the identity (11) by new integration variable $\lambda = z^{-1}$. Then we get

$$\int_1^{+\infty} \frac{z^a\, dz}{(1 + z^2)^2} = \int_0^{1} \frac{\lambda^{2-a}\, d\lambda}{(1 + \lambda^2)^2}.$$

Next, we substitute the above result into the right-hand side of the equality (11) and obtain

$$\int_0^{+\infty} \frac{z^a\, dz}{(1 + z^2)^2} = \int_0^{1} \frac{(z^a + z^{2-a})\, dz}{(1 + z^2)^2}. \tag{12}$$

The integral on the right-hand side of the equality (12) can be simplified by extracting a rational part. Using transformation

$$\frac{z^a + z^{2-a}}{(1 + z^2)^2} = \left\{ \frac{z^{1+a} + z^{3-a}}{2(1 + z^2)} \right\}' + \frac{(1 - a)(z^a - z^{2-a})}{2(1 + z^2)},$$

we obtain

$$\int_0^{1} \frac{(z^a + z^{2-a})\, dz}{(1 + z^2)^2} = \frac{z^{1+a} + z^{3-a}}{2(1 + z^2)} \bigg|_0^1 + \frac{1 - a}{2} \int_0^{1} \frac{(z^a - z^{2-a})\, dz}{1 + z^2}. \tag{13}$$

The first term on the right-hand side of the identity (13) shows that the considered integral (12) exists under the condition $|a - 1| < 2$. Thus, we get the formula

$$\int_0^{1} \frac{(z^a + z^{2-a})\, dz}{(1 + z^2)^2} = \frac{1}{2} + \frac{1 - a}{2} \int_0^{1} \frac{(z^a - z^{2-a})\, dz}{1 + z^2}. \tag{14}$$

Comparison of the results (12) and (14) allows us to obtain the identity

$$\int_0^{+\infty} \frac{z^a\, dz}{(1 + z^2)^2} = \frac{1}{2} + \frac{1 - a}{2} \int_0^{1} \frac{(z^a - z^{2-a})\, dz}{1 + z^2}. \tag{15}$$

The integral on the right-hand side of the equality (15) contains the integration variable z that varies from 0 to 1. This allows us to use the binomial series

$$(1+z^2)^{-1} = 1 + \sum_{k=1}^{+\infty} (-1)^k z^{2k}$$

to calculate the integral in the right-hand side of the identity (15). Then we have the equality

$$\int_0^1 \frac{(z^a - z^{2-a})\,dz}{1+z^2} = \int_0^1 \left(z^a - z^{2-a}\right) dz + \sum_{k=1}^{+\infty} (-1)^k \int_0^1 \left(z^{2k+a} - z^{2k+2-a}\right) dz,$$

which can also be represented in the following form

$$\int_0^1 \frac{(z^a - z^{2-a})\,dz}{1+z^2} = \sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{2k+1+a} - \frac{1}{2k+3-a}\right). \tag{16}$$

Let us introduce a new variable $k' = k+1$. Using it, we transform the sum in the right-hand side of the identity (16). We get the result

$$\int_0^1 \frac{(z^a - z^{2-a})\,dz}{1+z^2} = \sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{1-a-2k} + \frac{1}{1-a+2k}\right). \tag{17}$$

After that, we substitute the integral (17) into the right-hand side of the equality (15). Then we have

$$\int_0^{+\infty} \frac{z^a\,dz}{(1+z^2)^2} = \frac{1-a}{2} \left\{ \frac{1}{1-a} + \sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{1-a-2k} + \frac{1}{1-a+2k}\right) \right\}. \tag{18}$$

To calculate the sum of the series on the right-hand side of the identity (18), we use the so-called parameterization method (see, e.g., [15]), which is a powerful tool in mathematical physics. In this case, we can write the parameterizations

$$\frac{(-1)^k}{1-a-2k} = \frac{1}{2\cos\frac{\pi a}{2}} \int_0^{\pi} \cos\left\{ \left(\frac{1-a}{2} - k\right) t \right\} dt \tag{19}$$

and

$$\frac{(-1)^k}{1-a+2k} = \frac{1}{2\cos\frac{\pi a}{2}} \int_0^{\pi} \cos\left\{ \left(\frac{1-a}{2} + k\right) t \right\} dt. \tag{20}$$

Addition of the integral representations (19) and (20) gives us the following one

$$(-1)^k \left(\frac{1}{1-a-2k} + \frac{1}{1-a+2k}\right) = \frac{1}{\cos\frac{\pi a}{2}} \int_0^{\pi} \cos(kt) \cos\left\{ (1-a)\,\frac{t}{2} \right\} dt.$$

After summing over k in the above equality, we get the sum

$$\sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{1-a-2k} + \frac{1}{1-a+2k}\right) = \frac{1}{\cos\frac{\pi a}{2}} \int_0^{\pi} \left\{ \sum_{k=1}^{+\infty} \cos(kt) \right\} \cos\left\{ (1-a)\,\frac{t}{2} \right\} dt.$$

Let us add parameterization

$$\frac{1}{1-a} = \frac{1}{\cos\frac{\pi a}{2}} \int_0^{\pi} \left\{ \frac{1}{2} \right\} \cos\left\{ (1-a)\,\frac{t}{2} \right\} dt$$

to this sum. Therefore, we obtain

$$\frac{1}{1-a}+\sum_{k=1}^{+\infty}(-1)^k\left(\frac{1}{1-a-2k}+\frac{1}{1-a+2k}\right)$$
$$=\frac{1}{\cos\frac{\pi a}{2}}\int_0^{\pi}\left\{\frac{1}{2}+\sum_{k=1}^{+\infty}\cos(kt)\right\}\cos\left\{(1-a)\frac{t}{2}\right\}dt. \tag{21}$$

Next, we substitute the integral representation (21) into the formula (18). Thus, we get

$$\int_0^{+\infty}\frac{z^a\,dz}{(1+z^2)^2}=\frac{1-a}{2\cos\frac{\pi a}{2}}\int_0^{\pi}\left\{\frac{1}{2}+\sum_{k=1}^{+\infty}\cos(kt)\right\}\cos\left\{(1-a)\frac{t}{2}\right\}dt. \tag{22}$$

Evaluation of the integral on the right-hand side of the identity (22) becomes possible only after finding the sum of the series in curly braces. It should be noted that the sum of the considered series is a generalized function. This means that the classical methods of series summation are inapplicable in this case. To find the sum of the series, we use the well-known harmonic Poisson kernel

$$\frac{\sinh\varepsilon}{2(\cosh\varepsilon-\cos t)}=\frac{1}{2}+\sum_{k=1}^{+\infty}\exp(-\varepsilon k)\cos(kt),\quad \text{where } \varepsilon>0. \tag{23}$$

Let us transform the left-hand side of the equality (23). We use the double angle formulas for the functions $\sinh\varepsilon$, $\cosh\varepsilon$ and $\cos t$. This enables us to replace the identity (23) by the following one

$$\frac{\sinh\frac{\varepsilon}{2}\cosh\frac{\varepsilon}{2}}{2\left((\sinh\frac{\varepsilon}{2})^2+(\sin\frac{t}{2})^2\right)}=\frac{1}{2}+\sum_{k=1}^{+\infty}\exp(-\varepsilon k)\cos(kt). \tag{24}$$

Application of the parameterization

$$\frac{\sinh\frac{\varepsilon}{2}}{\left(\sinh\frac{\varepsilon}{2}\right)^2+\left(\sin\frac{t}{2}\right)^2}=\int_0^{+\infty}\exp\left(-q\sinh\frac{\varepsilon}{2}\right)\cos\left(q\sin\frac{t}{2}\right)dq$$

converts the formula (24) into the result

$$\frac{\cosh\frac{\varepsilon}{2}}{2}\int_0^{+\infty}\exp\left(-q\sinh\frac{\varepsilon}{2}\right)\cos\left(q\sin\frac{t}{2}\right)dq=\frac{1}{2}+\sum_{k=1}^{+\infty}\exp(-\varepsilon k)\cos(kt).$$

Now let $\varepsilon\to+0$. It follows from the above identity, that

$$\frac{1}{2}\int_0^{+\infty}\cos\left(q\sin\frac{t}{2}\right)dq=\frac{1}{2}+\sum_{k=1}^{+\infty}\cos(kt). \tag{25}$$

The equality (25) contains the integral representation

$$\delta\left(\sin\frac{t}{2}\right)=\frac{1}{\pi}\int_0^{+\infty}\cos\left(q\sin\frac{t}{2}\right)dq$$

for the Dirac delta function.

Then instead of the identity (25) we can write the result

$$\delta\left(\sin\frac{t}{2}\right)=\frac{2}{\pi}\left\{\frac{1}{2}+\sum_{k=1}^{+\infty}\cos(kt)\right\}$$

that allows us to reduce the integral (22) into the following form

$$\int_0^{+\infty} \frac{z^a\,dz}{(1+z^2)^2} = \frac{\pi(1-a)}{4\cos\frac{\pi a}{2}} \int_0^{\pi} \delta\left(\sin\frac{t}{2}\right)\cos\left\{(1-a)\,\frac{t}{2}\right\}dt. \tag{26}$$

Next, the integration variable $\xi = \sin(t/2)$ must be introduced in the integral (26). Then we obtain the following expression

$$\int_0^{+\infty} \frac{z^a\,dz}{(1+z^2)^2} = \frac{\pi(1-a)}{2\cos\frac{\pi a}{2}} \int_0^{1} \frac{\delta(\xi)\,d\xi}{\sqrt{1-\xi^2}} \cos\{\,(1-a)\arcsin(\xi)\,\}. \tag{27}$$

Using the well-known property $f(\xi)\,\delta(\xi-\xi') = f(\xi')\,\delta(\xi-\xi')$, from formula (27) we obtain

$$\int_0^{+\infty} \frac{z^a\,dz}{(1+z^2)^2} = \frac{\pi(1-a)}{2\cos\frac{\pi a}{2}} \int_0^{1} \delta(\xi)\,d\xi. \tag{28}$$

Consider another well-known property $\delta(\xi) = H'(\xi)$ represented via the Heaviside step function

$$H(\xi) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \frac{\sin(q\xi)}{q}\,dq = \begin{cases} 0, & \xi < 0, \\ 1/2, & \xi = 0, \\ 1, & \xi > 0. \end{cases}$$

Then we can write

$$\int_0^1 \delta(\xi)\,d\xi = \int_0^1 H'(\xi)\,d\xi = H(\xi)\Big|_0^1 = H(1) - H(0) = \frac{1}{2}. \tag{29}$$

Substituting the integral (29) into the expression (28), we obtain the result

$$\int_0^{+\infty} \frac{z^a\,dz}{(1+z^2)^2} = \frac{\pi(1-a)}{4\cos\frac{\pi a}{2}}. \tag{30}$$

If we apply the integral (30) to calculate the right-hand side of the identity (10), we get the result (7). The theorem is proved. □

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Бігармонійне рівняння в декартових координатах розглянуто для випадку верхньої півплощини. Розв’язок такого диференціального рівняння четвертого порядку в частинних похідних для заданих крайових умов подано у вигляді інтеграла від добутку функції та дельтаподібного ядра, який у даній роботі відіграє роль наближувального агрегату. Знайдено точну рівність для верхньої межі відхилення функцій класу Гельдера від розглядуваного бігармонійного оператора Пуассона в рівномірній метриці.

Ключові слова і фрази: бігармонійне рівняння, бігармонійний інтеграл Пуассона, клас Гельдера.