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Extending of partial metrics

Mykhaylyuk V.[1,2], Myronyk V.[2]

We investigate the following question: does there exist a compatible extension of a given compatible partial metric $p : A^2 \to \mathbb{R}$ on a closed subset A of a partially metrizable space X? We obtain a positive answer to this question in the case when the corresponding quasi-metric $q_p(x, y) = p(x, y) - p(x, x)$ has an extension that generates a weaker topology on X (in particular, if q_p is bounded). Moreover, we give an example which shows that in general the answer to the question is negative.

Key words and phrases: partial metric, quasi-metric, partially metrizable space, metrizable space, extension of metric, extension of quasi-metric, extension of partial metric, topological space.

[1] Jan Kochanowski University of Kielce, 5 Żeromskiego str., 25369, Kielce, Poland
[2] Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynskyi str., 58012, Chernivtsi, Ukraine
E-mail: vmykhaylyuk@ukr.net (Mykhaylyuk V.), vadmyron@gmail.com (Myronyk V.)

Introduction

The notion of a partial metric space, as a certain weakening of the notion of metric space, was introduced by S.G. Matthews [13] in 1992. This notion is quite widely used in research from the Fixed Point Theory (see, for example, [9] and the literature given there). At the same time, the topological and metric properties of partial metric spaces are studied [5, 11, 12]. In connection with this, the following question naturally arises: which of the results of the theory of metric spaces are transferred without change to the case of partial metric spaces or have their own analogues? It was proved in [16] that similarly to metrizable spaces, compactness, countable compactness, and sequential compactness are equivalent for the class of partially metrizable spaces. Moreover, necessary and sufficient conditions for the metrizability of partial metric spaces are established in [17].

In [6], the following well-known result was proved by F. Hausdorff (see also [3,7,8]).

Theorem ([6])**.** *Let X be a metrizable space, $A \subseteq X$ be a closed subset and $d_A : A^2 \to \mathbb{R}$ be a compatible metric on A. Then there exists a compatible metric $d : X^2 \to \mathbb{R}$ on X such that $d(x, y) = d_A(x, y)$ for every $x, y \in A$.*

Notice that this result was developed and generalized by many mathematicians (see, for example, [1, 2, 18, 20] and the literature given there).

This article is devoted to the research of the following question.

Question 1. *Let X be a partially metrizable space, $A \subseteq X$ be a closed subset of X and $p : A^2 \to \mathbb{R}$ be a compatible partial metric on A. Does there exist a compatible partial metric $\tilde{p}$ on X such that $\tilde{p}(x, y) = p(x, y)$ for every $x, y \in A$?*

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First, we investigate a similar question on the extending of a given quasi-pseudometric q from a given closed subset A of a quasi-metrizable space X to the entire space X. In Section 2, we apply Bing's approach to the proof of Hausdorff's Theorem from [3] and prove the possibility of such an extension of q in the case when q is upper bounded by some compatible quasi-pseudometric on X. In Section 3, we introduce a notion of weakly compatible quasi-pseudometrics and show that the existence of a compatible extension of q is equivalent to the existence of a weakly compatible extension q. In particular, it implies that every bounded quasi-pseudometric has a compatible extension. In Section 4, we obtain a positive answer to Question 1 in the case when the corresponding quasi-metric q_p has a weakly compatible extension (in particular, if q_p is bounded) using the result on the extending of quasi-metric. In Section 5, we give an example that shows the essentiality of the existence of a weakly compatible extension for extending of quasi-pseudometric and partial metric. This example, in particular, shows that in general the answer to Question 1 is negative.

1 Basic notions and denotations

A function $q : X^2 \to [0, +\infty)$ is called *a quasi-pseudometric* on a set X (see [10, 19]) if

(q_1) $q(x, x) = 0$,

(q_2) $q(x, z) \le q(x, y) + q(y, z)$

for all $x, y, z \in X$. A pair (X, q), where X is a set and q is a quasi-pseudometric on X, is called a *quasi-pseudometric space*.

Let (X, q) be a quasi-pseudometric space. For every $x \in X$ the balls

$$B_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}, \quad \varepsilon > 0,$$

form a base of the *quasi-pseudometric topology* τ_q at the point x.

A quasi-pseudometric $q : X^2 \to [0, +\infty)$ is called *a quasi-metric* on X (see [14]) if

(q_3) $x = y \Leftrightarrow q(x, y) = q(y, x) = 0$

for every $x, y \in X$ (this means that (X, q) is a T_0-space); and *an asymmetric metric* on X (see [4]) if

(q_4) $x = y \Leftrightarrow q(x, y) = 0$

for every $x, y \in X$ (this means that (X, q) is a T_1-space).

A function $p : X^2 \to [0, +\infty)$ is called *a partial metric* on X (see [14]) if

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \le p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$

for all $x, y, z \in X$. A pair (X, p), where X is a set and p is a quasi-pseudometric on X, is called a *partially metric space*.

For any partial metric $p : X^2 \to [0, +\infty)$ the function $q_p : X^2 \to [0, +\infty)$ defined by

$$q_p(x, y) = p(x, y) - p(x, x), \quad (x, y) \in X^2,$$

is a quasi-metric on X and the topology of the partial metric space (X, p) is the topology of the quasi-metric space (X, q_p) (see [14, Theorem 4.1]).

Moreover, the function $d_p : X^2 \to [0, +\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \quad (x,y) \in X^2,$$

is a metric on X.

Quasi-pseudometrics p and q on a set X are called *equivalent* if the topologies τ_p and τ_q coincide.

Let X be a topological space and $A \subseteq X$. We say that a quasi-pseudometric (partial metric) p on A is *compatible* if p generates on A the topology of the subspace A of X. A topological space X is *quasi-pseudometrizable (quasi-metrizable, partially metrizable)* if X has a compatible quasi-pseudometric (quasi-metric, partial metric) on X.

Let $A \subseteq X$, $p : X^2 \to [0, +\infty)$ and $q : A^2 \to [0, +\infty)$ be quasi-metrics (partial metrics). We say that p is an *extension of* q if $p(x,y) = q(x,y)$ for every $x, y \in A$.

2 Extending of bounded quasi-pseudometric

We can prove the following proposition analogously as [3, Theorem 5] (see also [8, Theorem 24]).

Proposition 1. *Let (X, r) be a quasi-pseudometric space, $A \subseteq X$ and (A, q) be a quasi-pseudometric space such that*

- (i) *A is τ_r-closed,*
- (ii) *$q(x,y) \le r(x,y)$ for every $x, y \in A$,*
- (iii) *the quasi-pseudometrics r and q are equivalent on A.*

Then there exists a quasi-pseudometric $\tilde{q}$ on X such that

- (α) *$\tilde{q}$ is an extension of q,*
- (β) *$\tilde{q}(x,y) \le r(x,y)$ for every $x, y \in X$,*
- (γ) *the quasi-pseudometrics r and $\tilde{q}$ are equivalent on X.*

Proof. Consider the following function $\tilde{q} : X^2 \to \mathbb{R}$, defined by

$$\tilde{q}(x,y) = \min\Big\{ r(x,y), \inf_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,y) \big) \Big\}.$$

Clearly, $\tilde{q}$ satisfies (β). Moreover, since the quasi-pseudometrics q and r satisfy (ii), we have

$$\tilde{q}(x,y) = \min\Big\{ r(x,y), \inf_{b \in A} \big(q(x,b) + r(b,y) \big) \Big\} = \inf_{b \in A} \big(q(x,b) + r(b,y) \big)$$

for every $x \in A$ and $y \in X$,

$$\tilde{q}(x,y) = \min\Big\{ r(x,y), \inf_{a \in A} \big(r(x,a) + q(a,y) \big) \Big\} = \inf_{a \in A} \big(r(x,a) + q(a,y) \big)$$

for every $x \in X$ and $y \in A$,

$$\tilde{q}(x,y) = q(x,y)$$

for every $x, y \in A$. That is, $\tilde{q}$ satisfies (α).

Show that $\tilde{q}$ is a quasi-pseudometric on X. The condition (q_1) is obvious. Let us show (q_2). Let $x, y, z \in X$. We consider the following four cases.

Case 1. Let $\tilde{q}(x,y) = r(x,y)$ and $\tilde{q}(y,z) = r(y,z)$. Then

$$\tilde{q}(x,y) + \tilde{q}(y,z) = r(x,y) + r(y,z) \geq r(x,z) \geq \tilde{q}(x,z).$$

Case 2. Let $\tilde{q}(x,y) = r(x,y)$ and $\tilde{q}(y,z) = \inf\limits_{a,b \in A} \big(r(y,a) + q(a,b) + r(b,z)\big)$. Then

$$\begin{aligned}\tilde{q}(x,y) + \tilde{q}(y,z) &= \inf_{a,b \in A} \big(r(x,y) + r(y,a) + q(a,b) + r(b,z)\big) \\ &\geq \inf_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,z)\big) \geq \tilde{q}(x,z).\end{aligned}$$

Case 3. Let $\tilde{q}(x,y) = \inf\limits_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,y)\big)$ and $\tilde{q}(y,z) = r(y,z)$. Then

$$\begin{aligned}\tilde{q}(x,y) + \tilde{q}(y,z) &= \inf_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,y) + r(y,z)\big) \\ &\geq \inf_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,z)\big) \geq \tilde{q}(x,z).\end{aligned}$$

Case 4. Let

$$\tilde{q}(x,y) = \inf_{a,b \in A} \big(r(x,a) + q(a,b) + r(b,y)\big)$$

and

$$\tilde{q}(y,z) = \inf_{a,b \in A} \big(r(y,a) + q(a,b) + r(b,z)\big).$$

Then according to (ii), we have

$$\begin{aligned}\tilde{q}(x,y) + \tilde{q}(y,z) &= \inf_{a,b,c,d \in A} \big(r(x,a) + q(a,b) + r(b,y) + r(y,c) + q(c,d) + r(d,z)\big) \\ &\geq \inf_{a,d \in A} \big(r(x,a) + q(a,d) + r(d,z)\big) \geq \tilde{q}(x,z).\end{aligned}$$

What is left is to show (γ). According to (β), we have $\tau_{\tilde{q}} \subseteq \tau_r$. It remains to verify that $\tau_{\tilde{q}} \supseteq \tau_r$. Let $x_0 \in X$ and $\varepsilon > 0$. To show that there exists $\delta > 0$ such that $B_{\tilde{q}}(x_0, \delta) \subseteq B_r(x_0, \varepsilon)$ we consider the following two cases.

Case A. Let $x_0 \in A$. According to (iii), there exists $\delta_1 > 0$ such that for every $x \in A$ we have

$$q(x_0, x) < \delta_1 \;\Rightarrow\; r(x_0, x) < \frac{\varepsilon}{2}.$$

We set $\delta = \min\left\{\delta_1, \frac{\varepsilon}{2}\right\}$. Let $x \in B_{\tilde{q}}(x_0, \delta)$. Since $\tilde{q}(x_0, x) = \inf\limits_{a \in A} \big(r(x_0, a) + r(a, x)\big)$, there exists an $a \in A$ such that

$$q(x_0, a) + r(a, x) < \delta.$$

Then $q(x_0, a) < \delta_1$ and $r(a, x) < \frac{\varepsilon}{2}$. Therefore,

$$r(x_0, x) \leq r(x_0, a) + r(a, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Case B. Let $x_0 \in X \setminus A$. Since A is τ_r-closed, there exists a positive number $\delta \leq \varepsilon$ such that $B_r(x_0, \delta) \cap A = \varnothing$. Notice that

$$\tilde{q}(x_0, x) = \inf_{a \in A} \big(r(x_0, a) + q(a, x)\big) \geq \delta$$

for every $x \in A$. Then for every $x \in B_{\tilde{q}}(x_0, \delta)$ we have

$$\varepsilon \geq \delta > \tilde{q}(x_0, x) = \min\left\{r(x_0, x),\, \inf_{a,b \in A} \big(r(x_0, a) + q(a,b) + r(b,x)\big)\right\} = r(x_0, x).$$

□

3 Weakly compatible and bounded quasi-pseudometric

In this section, we investigate the existence of a compatible quasi-pseudometric r on a quasi-pseudometrizable space, which for a given quasi-pseudometric q satisfies condition (ii) from Proposition 1.

We say that a quasi-pseudometric q on a topological space (X, τ) is *weakly compatible* if $\tau_q \subseteq \tau$.

Proposition 2. *Let X be a quasi-pseudometrizable space, $A \subseteq X$ be a closed subset of X and q be a bounded compatible quasi-pseudometric on A. Then there exists a weakly compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q(x,y)$ for every $x, y \in A$.*

Proof. Choose $C > 0$ such that $q(x) \leq C$ for every $x \in A$ and consider the function $\tilde{q} : X^2 \to \mathbb{R}$, defined by

$$\tilde{q}(x,y) = \begin{cases} q(x,y), & \text{if } x, y \in A, \\ 0, & \text{if } x \in X \text{ and } y \in X \setminus A, \\ C, & \text{if } x \in X \setminus A \text{ and } y \in A. \end{cases}$$

Clearly, $\tilde{q}$ is an extension of q that satisfies (q_1). Show that $\tilde{q}$ is a quasi-pseudometric on X, that is, $\tilde{q}$ satisfies (q_2). Let $x, y, z \in X$. We consider the following four cases.

If $z \in X \setminus A$, then $\tilde{q}(x,z) = 0 \leq \tilde{q}(x,y) + \tilde{q}(y,z)$.

If $z \in A$ and $y \in X \setminus A$, then $\tilde{q}(x,y) + \tilde{q}(y,z) \geq \tilde{q}(y,z) = C \geq \tilde{q}(x,z)$.

If $y \in A$ and $x \in X \setminus A$, then $\tilde{q}(x,y) + \tilde{q}(y,z) \geq \tilde{q}(x,y) = C \geq \tilde{q}(x,z)$.

Finally, if $x, y, z \in A$, then $\tilde{q}(x,y) + \tilde{q}(y,z) = q(x,y) + q(y,z) \geq q(x,z) = \tilde{q}(x,z)$.

Let τ be the topology of X. Since q is compatible on the τ-closed set A and $\tilde{q}(x,y) = 0$ for every $x \in X$ and $y \in X \setminus A$, we have

$$\tau_{\tilde{q}} = \{(X \setminus A) \cup B : B \in \tau_q\} \cup \{\varnothing\} \subseteq \tau.$$

Thus, $\tilde{q}$ is weakly compatible on X. □

Proposition 3. *Let p and q be quasi-pseudometrics on a set X such that $\tau_p \subseteq \tau_q$. Then the function $r = p + q$ is a quasi-pseudometric on X such that $\tau_r = \tau_q$.*

Proof. Clearly, r is a quasi-pseudometric on X. Moreover, since $q \leq r$, we get $\tau_q \subseteq \tau_r$. It remains to show that $\tau_r \subseteq \tau_q$. Let $x_0 \in X$ be a fixed point, $\varepsilon > 0$ and $U = \{x \in X : r(x_0, x) < \varepsilon\}$. Consider the τ_q-neighbourhood $U_1 = \{x \in X : q(x_0, x) < \frac{\varepsilon}{2}\}$ of x_0 and τ_p-neighbourhood $U_2 = \{x \in X : p(x_0, x) < \frac{\varepsilon}{2}\}$ of x_0. Since $\tau_p \subseteq \tau_q$, $U_1 \cap U_2$ is a τ_q-neighbourhood of x_0. Moreover, $U_1 \cap U_2 \subseteq U$. Thus, U is a τ_q-neighbourhood of x_0 and $\tau_r \subseteq \tau_q$. □

Proposition 4. *Let X be a quasi-pseudometrizable space, $A \subseteq X$ be a closed subset of X and q be a bounded compatible quasi-pseudometric on A. Then the following conditions are equivalent:*

(i) *there exists a compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q(x,y)$ for every $x, y \in A$,*

(ii) *there exists a weakly compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q(x,y)$ for every $x, y \in A$.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$. Let q_1 be a compatible quasi-pseudometric and q_2 be a weakly compatible quasi-pseudometric on X such that $q_2(x,y) = q(x,y)$ for every $x, y \in A$. By Proposition 3 the function $r = q_1 + q_2$ is a compatible quasi-pseudometric on X. Moreover, $q(x,y) \le r(x,y)$ for every $x, y \in A$. According to Proposition 1, there exists a compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q(x,y)$ for every $x, y \in A$. □

Corollary 1. *Let X be a quasi-pseudometrizable space, $A \subseteq X$ be a closed subset of X and q be a bounded compatible quasi-pseudometric on A. Then there exists a compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q(x,y)$ for every $x, y \in A$.*

4 Extending of partial metric

In this section, we obtain the main result of our paper, which gives a positive answer to Question 1 using Corollary 1 and the following well-known McShane's result on the extending of a Lipschitz function.

Proposition 5 ([15, Theorem 1])**.** *Let (X,d) be a metric space, $A \subseteq X$ be a subset and $f : A \to \mathbb{R}$ be a Lipschitz function with a constant $C \ge 0$. Then there exists a Lipschitz function $g : X \to \mathbb{R}$ with the constant C such that $g(x) = f(x)$ for every $x \in A$.*

Proposition 6. *Let (X,p) be a partial metric space. Then the function $f : X \to \mathbb{R}$, defined by $f(x) = p(x,x)$, is a 1-Lipschitz function with respect to the metric d_p.*

Proof. The statement follows immediately from the next inequality

$$f(x) - f(y) = p(x,x) - p(y,y) = d_p(x,y) - 2\big(p(x,y) - p(x,x)\big) \le d_p(x,y).$$

□

Proposition 7. *Let (X,d) be a metric space and $f : X \to [0,+\infty)$ be a 1-Lipschitz function. Then the function $p : X^2 \to \mathbb{R}$, defined by*

$$p(x,y) = \frac{1}{2}\big(d(x,y) + f(x) + f(y)\big),$$

is a partial metric on X such that $d = d_p$ and $p(x,x) = f(x)$ for every $x \in X$.

Proof. Clearly, $p(x,x) = f(x)$ for every $x \in X$. It remains to verify $(p_1) - (p_4)$.

(p_1) The following implications

$$x = y \quad \Leftrightarrow \quad d(x,y) = 0 \quad \Leftrightarrow \quad p(x,x) = p(x,y) = p(y,y)$$

are obvious.

(p_2) Since f is 1-Lipschitz, $f(y) \ge f(x) - d(x,y)$ for every $x, y \in X$. Then

$$p(x,y) - p(x,x) = \frac{1}{2}\big(d(x,y) - f(x) + f(y)\big) \ge \frac{1}{2}\big(d(x,y) - f(x) + f(x) - d(x,y)\big) = 0.$$

Condition (p_3) is obvious.

(p_4) For every $x, y, z \in X$ we have

$$\begin{aligned} p(x,y) + p(y,z) &= \frac{1}{2}\big(d(x,y) + f(x) + f(y) + d(y,z) + f(y) + f(z)\big) \\ &\ge \frac{1}{2}\big(d(x,z) + f(x) + f(z)\big) + f(y) = p(x,z) + p(y,y). \end{aligned}$$

□

The following theorem is the main result of the paper.

Theorem 2. *Let X be a partial metrizable space, $A \subseteq X$ be a closed subset of X and $p : A^2 \to \mathbb{R}$ be a compatible partial metric on A such that the quasi-metric q_p is bounded. Then there exists a compatible partial metric $\tilde{p} : X^2 \to \mathbb{R}$ on X such that $\tilde{p}(x,y) = p(x,y)$ for every $x, y \in A$.*

Proof. It follows from Corollary 1 that there exists a compatible quasi-pseudometric $\tilde{q}$ on X such that $\tilde{q}(x,y) = q_p(x,y)$ for every $x, y \in A$. Since X is partial metrizable, X is a T_0-space. Therefore, the compatible quasi-pseudometric $\tilde{q}$ is a quasi-metric.

Consider the metric $d : X^2 \to \mathbb{R}$, defined by $d(x,y) = \tilde{q}(x,y) + \tilde{q}(y,x)$. Clearly, d is an extension of d_p. It follows from Proposition 6 that the function $f : (A, d_p) \to \mathbb{R}$, defined by $f(x) = p(x,x)$, is an 1-Lipschitz function. According to Proposition 5, there exists an 1-Lipschitz function $\tilde{f} : (X,d) \to [0,+\infty)$ such that $\tilde{f}|_A = f$. It remains to consider the function $\tilde{p} : X^2 \to \mathbb{R}$, defined by

$$\tilde{p}(x,y) = \frac{1}{2}\big(d(x,y) + \tilde{f}(x) + \tilde{f}(y)\big),$$

which is a partial metric according to Proposition 7. Clearly, $\tilde{p}$ is an extension of p. Moreover, since $q_{\tilde{p}} = \tilde{q}$, $\tilde{p}$ is compatible on X. □

5 Examples and questions

The following example shows the essentiality of the existence of a weakly compatible extension in Theorem 2 and Corollary 1. This example shows that, in general, the answer to Question 1 is negative.

Proposition 8. *There exist a partial metric space (X, p), a τ_{q_p}-closed set $A \subseteq X$ and a partial metric r on A such that*

(1) q_p and q_r are equivalent on A,

(2) $\tau_{q_s} \not\subseteq \tau_{q_p}$ for every extension s of r on X.

Proof. Let $X = \{x_n : n = 0, 1, 2, \ldots\}$ be any countable set, where $x_n \neq x_k$ for any distinct n, k, and let $p : X^2 \to \mathbb{R}$ be a function defined by

$$p(x,y) = p(y,x) = \begin{cases} 0, & \text{if } x = y = x_0, \\ 1, & \text{if } x = y \neq x_0, \\ 1, & \text{if } x \neq y = x_0, \\ 2, & \text{otherwise.} \end{cases}$$

We show that p is a partial metric on X. It is enough to verify condition (p_4) for the case of distinct $x, y, z \in X$. If $y = x_0$ then

$$p(x,y) + p(y,z) = 1 + 1 = p(x,z) = p(x,z) + p(y,y).$$

If $y \neq x_0$ then $p(x,y) = 2$ or $p(y,z) = 2$. Therefore,

$$p(x,y) + p(y,z) \geq 2 + 1 \geq p(x,z) + p(y,y).$$

Thus, p satisfies (p_4) and p is a partial metric on X.

Notice that

$$q_p(x,y) = \begin{cases} 0, & \text{if } x = y \text{ or } y = x_0, \\ 1, & \text{otherwise.} \end{cases}$$

Consider the set $A = \{x_n : n \in \mathbb{N}\}$. Clearly, A is a τ_{q_p}-closed discrete subset of X.

Consider the metric $r : A^2 \to \mathbb{R}$ defined by

$$r(x_n, x_k) = |n - k|.$$

It is obvious that $q_r = r$ and q_p are equivalent on A. Assume that a partial metric $s : X^2 \to \mathbb{R}$ is an extension of r such that $\tau_{q_s} \subseteq \tau_{q_p}$. Then q_s is an extension of $q_r = r$ on X. Since $x_0 \in G$ for every nonempty $G \in \tau_{q_p}$, $x_0 \in G$ for every nonempty $G \in \tau_{q_s}$. Therefore, $q_s(x_n, x_0) = 0$ for every $n \in \mathbb{N}$ and

$$q_s(x_0, x_1) = q_s(x_n, x_0) + q_s(x_0, x_1) \geq q_s(x_n, x_1) = r(x_n, x_1) = n - 1$$

for every $n \in \mathbb{N}$, a contradiction. □

Corollary 2. *There exists a quasi-metrizable space X, a closed subset A of X and a compatible quasi-metric q on A such that q cannot be extended to a compatible quasi-metric on X.*

Proof. It is enough to consider the space X, the set A and the quasi-metric $q = q_r$ from Proposition 8. □

Notice that the partial metrizable space X from Proposition 8 is not a T_1-space, that is, X has no compatible asymmetric metric. Therefore, the following questions naturally arise.

Question 3. *Let X be a partially metrizable T_1-space, $A \subseteq X$ be a closed subset of X and $p : A^2 \to \mathbb{R}$ be a compatible partial metric on A. Does there exist a compatible partial metric $\tilde{p}$ on X which is an extension of p?*

Question 4. *Let X be a quasi-metrizable T_1-space, $A \subseteq X$ be a closed subset of X and $q : A^2 \to \mathbb{R}$ be a compatible quasi-metric on A. Does there exist a compatible partial metric $\tilde{q}$ on X which is an extension of q?*

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Михайлюк В., Мироник В. *Продовження часткових метрик* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 33–41.

У статті досліджується таке питання: чи кожну часткову метрику $p : A^2 \to \mathbb{R}$, яка визначена на замкненій підмножині A частково метризовного простору X і узгоджена з його топологією на A, можна продовжити на весь простір зі збереженням топологічної структури? Отримано позитивну відповідь на це питання у випадку, коли відповідна квазіметрика $q_p(x,y) = p(x,y) - p(x,x)$ має продовження, яке породжує слабшу топологію на просторі X (зокрема, якщо q_p обмежена). Крім того, побудовано приклад, який у загальному випадку дає негативну відповідь на сформульоване вище питання.

Ключові слова і фрази: часткова метрика, квазіметрика, частково метризовний простір, метризовний простір, продовження метрики, продовження квазіметрики, продовження часткової метрики, топологічний простір.