ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2025, **17** (1), 5–13
doi:10.15330/cmp.17.1.5-13

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.5–13

Extension property for equi-Lebesgue families of functions

Karlova O.[1,2]

Let X be a topological space and (Y, d) be a complete separable metric space. For a family $\mathscr{F}$ of functions from X to Y we say that $\mathscr{F}$ is equi-Lebesgue if for every $\varepsilon > 0$ there is a cover (F_n) of X consisting of closed sets such that $\operatorname{diam} f(F_n) \le \varepsilon$ for all $n \in \mathbb{N}$ and $f \in \mathscr{F}$.

We prove that if X is a perfectly normal space, Y is a complete separable metric space and $E \subseteq X$ is an arbitrary set, then every equi-continuous family $\mathscr{F} \subseteq Y^E$ can be extended to an equi-Lebesgue family $\mathscr{G} \subseteq Y^X$.

Key words and phrases: extension of Borel 1 function, equi-Baire 1 family of functions, equi-Lebesgue family of functions, 1-separated set, metrizable space, topological space.

[1] Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynskyi str., 58012, Chernivtsi, Ukraine
[2] Jan Kochanowski University of Kielce, 5 Żeromskiego str., 25369, Kielce, Poland
E-mail: o.karlova@chnu.edu.ua

1 Introduction

Recall that a function $f : X \to Y$ between topological spaces X and Y is

(i) *Baire 1*, if f is a pointwise limit of a sequence of continuous functions $f_n : X \to Y$;

(ii) *Borel 1* or *F_σ-measurable*, if for each open set $V \subseteq Y$ the preimage $f^{-1}(V)$ is F_σ in X.

We will denote by $B_1(X, Y)$ and $\mathscr{B}_1(X, Y)$ the collections of all Baire 1 and Borel 1 functions, respectively.

It is well-known that for a perfectly normal (in particular, metric) topological space X and for a metric space Y every Baire 1 function is F_σ-measurable; moreover, for $Y = \mathbb{R}$ these two notions are equivalent [10]. But $\mathscr{B}_1(X, Y) \not\subseteq B_1(X, Y)$ even for metric complete separable spaces X and Y as the following simple example shows: $\chi_{\{0\}} \in \mathscr{B}_1(\mathbb{R}, \mathbb{R}) \setminus B_1(\mathbb{R}, \mathbb{R})$.

Many authors use the term *Baire 1* for functions between topological spaces in the sense of F_σ-measurable function. We prefer to use notion *Borel 1* instead of *Baire 1* in such cases and throughout the paper we will cite results of other authors using this terminology.

In 2001, P.Y. Lee, W.-K. Tang and D. Zhao [13] obtained the following ε-δ characterization of Borel 1 functions.

УДК 515.1, 517.51
2020 *Mathematics Subject Classification:* Primary 54C20, 26A21; Secondary 54C30, 54C50.

Author want to thanks professor Piotr Szuca for his hints about applications of the orbit function for family of equi-Lebesgue maps.

Theorem 1. *Let (X, d_X) be a separable metric space and (Y, d_Y) be a complete separable metric space. A function $f\colon X \to Y$ is Borel 1 if and only if for each $\varepsilon > 0$ there exists a function $\delta_\varepsilon^f\colon X \to (0, +\infty)$ such that for all $x, x' \in X$ we have*

$$d_X(x, x') < \min\left\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(x')\right\} \implies d_Y(f(x), f(x')) \le \varepsilon. \tag{1}$$

Motivated by this characterization, D. Lecomte [12] introduced the notion of equi-Baire 1 family of functions, which was rediscovered later by A. Alikhani-Koopaei [1]. Namely, a family $\mathscr{F}$ of functions from X to Y is said to be *equi-Baire 1* if for each $\varepsilon > 0$ there exists a function $\delta_\varepsilon\colon X \to (0, +\infty)$ such that for all $f \in \mathscr{F}$ and $x, x' \in X$ condition (1) holds.

D. Lecomte [12, Proposition 32] obtained the following characterization of equi-Baire 1 families.

Theorem 2. *Let (X, d_X) be a separable metric space and (Y, d_Y) be a complete separable metric space. For a family $\mathscr{F}$ of functions from X to Y the following conditions are equivalent:*

(i) *$\mathscr{F}$ is equi-Baire 1;*

(ii) *for every $\varepsilon > 0$ there is a cover (F_n) of X consisting of closed sets such that*
$$\operatorname{diam} f(F_n) \le \varepsilon$$
for all $n \in \mathbb{N}$ and $f \in \mathscr{F}$;

(iii) *there is a finer metrizable separable topology on X making $\mathscr{F}$ equi-continuous;*

(iv) *for every nonempty closed subset F of X there is a point x such that the family*
$$\{f|_F : f \in \mathscr{F}\}$$
is equi-continuous at x.

Properties of equi-Baire 1 families of functions and its applications for dynamic systems were studied recently in [1–4].

In [3], the authors introduced *equi-Lebesgue families* of functions as families with property *(ii)* from Theorem 2. One of the main results of [3] deals with an extension property of equi-Lebesgue families.

Theorem 3 ([3, Theorem 6.1])**.** *Let (X, d_X) be a separable metric space and (Y, d_Y) be a separable complete metric space. Let $H \subset X$ be a nonempty G_δ-set and $\mathscr{F}$ be an equi-continuous family of functions from H to (Y, d_Y). Then all functions in $\mathscr{F}$ can be extended to an equi-Baire 1 family of functions from X to Y.*

The aim of this note is a generalization of Theorem 3. Namely, we prove the following fact.

Theorem 4. *Let X be a perfectly normal space, Y be a Polish space and $E \subseteq X$ be an arbitrary set. Then every equi-continuous family $\mathscr{F} \subseteq Y^E$ can be extended to equi-Lebesgue family $\mathscr{G} \subseteq Y^X$.*

The paper is organized as follows. In Section 2, using standard arguments, we show that every equi-continuous family $\mathscr{F} \subseteq Y^H$ can be extended to an equi-continuous family $\mathscr{G} \subseteq Y^E$ for some G_δ-set $E \supseteq H$ in X. Later we consider 1-separated sets in Section 3 and prove that in a perfectly normal space every hereditarily Baire subset is 1-separated from any disjoint G_δ-set. This gives a possibility to extend Borel functions from hereditarily Baire subsets of perfectly normal spaces. We prove this in Section 4. Finally, Section 5 contains the proof of the main extension theorem of the paper.

2 Extension of equi-continuous family to a G_δ-set

Let X be a topological space and (Y, d) be a metric space. For a function $f : X \to Y$ we consider the following property:

(LP) for every $\varepsilon > 0$ there is a sequence (F_n) of closed sets in X such that $X = \bigcup_{n=1}^{\infty} F_n$ and $\operatorname{diam} f(F_n) < \varepsilon$ for every $n \in \mathbb{N}$.

In case $X = Y = \mathbb{R}$, H. Lebesgue proved [11] that the above mentioned condition is equivalent to the inclusion $f \in \mathscr{B}_1(X, Y)$. In [3], this property of a function is called *Lebesgue property*.

In is known (see [10, §31.II, Theorem 3]), that every function with *(LP)* is Borel 1, and if Y is separable, then the inverse implication is true. It was shown in [3], that the condition of separability on Y is essential.

If a function f between metric spaces X and Y satisfies condition (1), then we will say, following [3], that f *has LTZ-property*.

Let us recall that if a single-function family $\mathscr{F} = \{f\}$ has property *(iv)* of Theorem 2, then we say that f has the *point of continuity property* or, briefly, *(PCP)*. Similarly, a family $\mathscr{F}$ having *(iv)* is called *a family with the point of equi-continuity property* or *(PECP)* for short.

Let X be a topological space and (Y, d) be a bounded metric space. For a family $\mathscr{F} \subseteq Y^X$ of functions we denote by

$$f^\sharp_{\mathscr{F}}(x) = \big(f(x)\big)_{f \in \mathscr{F}}$$

the *orbit function* $f^\sharp_{\mathscr{F}} : X \to Y^T$, where $T = |\mathscr{F}|$. Assume that $Z = Y^T$ is equipped with the supremum metric

$$\varrho(z_1, z_2) = \sup_{t \in T} d\left(z_1(t), z_2(t)\right).$$

Then it is easy to see that the following observation is valid.

Proposition 1. *Let X be a topological space and (Y, d) be a bounded metric space. Then*

(1) $\mathscr{F}$ is equi-continuous at $x \in X$ if and only if $f^\sharp_{\mathscr{F}} : X \to (Z, \varrho)$ is continuous at x;

(2) $\mathscr{F}$ is equi-Lebesgue if and only if $f^\sharp_{\mathscr{F}} : X \to (Z, \varrho)$ has Lebesgue property;

(3) $\mathscr{F}$ has (PECP) if and only if $f^\sharp_{\mathscr{F}} : X \to (Z, \varrho)$ has (PCP);

(4) if X is metric, then $\mathscr{F}$ is equi-Baire 1 if and only if $f^\sharp_{\mathscr{F}} : X \to (Z, \varrho)$ has LTZ-property;

Definition 1. *Let $A \subseteq X$. We say that a family $\mathscr{G} \subseteq Y^X$ is an extension of a family $\mathscr{F} \subseteq Y^A$ if for every $f \in \mathscr{F}$ there is $g \in \mathscr{G}$ such that $g|_A = f$.*

Let us recall that a topological space is *perfect*, if every its closed subset is G_δ.

Proposition 2. *Let X be a perfect topological space, (Y, d) be a complete bounded metric space, $H \subseteq X$ be an arbitrary set and $\mathscr{F} \subseteq Y^H$ be an equi-continuous family of functions. Then $\mathscr{F}$ can be extended to an equi-continuous family $\mathscr{G} \subseteq Y^E$ onto a G_δ-set $E \supseteq H$.*

Proof. Let $\mathscr{F} \subseteq Y^H$ be an equi-continuous family of functions $\mathscr{F} = \{f_t : t \in T\}$. Then $f^\sharp_{\mathscr{F}} : H \to (Z, \varrho)$ is continuous on H. Since the space (Z, ϱ) is complete, it follows from [5, 4.3.16] that there exists a continuous extension $g : E \to (Z, \varrho)$ of $f^\sharp_{\mathscr{F}}$, where $E = \omega_g^{-1}(0)$. Let $g(x) = (g_t(x))_{t \in T}$ for each $x \in E$. Then family $\mathscr{G} = \{g_t : t \in T\}$ is an equi-continuous extension of $\mathscr{F}$ by Proposition 1. Note that the oscillation function $\omega_g : E \to \mathbb{R}$ is upper semi-continuous, consequently, E is closed in X. Moreover, E is a G_δ-subset of a perfect space X. □

3 1-separated sets in a perfectly normal paracompact space

In this section, we deal with a notion of 1-separated subsets which plays crucial role in extension of Borel 1 functions.

Definition 2. *Subsets A and B in a topological space X are called 1-separated, if there exists an F_σ- and G_δ-set $H \subseteq X$ such that*

$$A \subseteq H \subseteq X \setminus B.$$

In this case, we say that H separates A and B.

Remark 1. *Let X be a perfectly normal space.*

- *Definition 2 is equivalent to the definition of 1-separated sets from [8].*
- *If A and B are disjoint G_δ-subsets of X, then they are 1-separated [10, §30, Theorem 2].*

Definition 3. *Let us recall that a set $A \neq \varnothing$ in a topological space X is reducible (in the sense of Hausdorff), if for every closed set $F \neq \varnothing$ we have*

$$\overline{F \cap A} \cap \overline{F \setminus A} \neq F.$$

Recall that a topological space is *hereditarily Baire*, if every its closed subset is a Baire space.

Clearly, each open or closed set is reducible. Notice that every reducible subset of a perfectly normal paracompact space is F_σ and G_δ simultaneously (see [7, Theorem 1]). Moreover, if X is hereditarily Baire, the inverse is true [7, Proposition 3.1].

Definition 4. *Let $\mathscr{D} = \{D_\xi : \xi \in [0, \alpha]\}$ be an ordinal-indexed family of closed subsets of a topological space X. Family $\mathscr{D}$ is said to be regular closed in X, if*

(a) $D_0 = X \supset D_1 \supset \cdots \supset D_\alpha = \varnothing$;

(b) $D_\gamma = \bigcap_{\xi<\gamma} D_\xi$ if $\gamma \in [0, \alpha]$ is limit.

By [9, Lemma 2.2] the following property holds.

Proposition 3. *Let X be a topological space and $A \subseteq X$.*
The following conditions are equivalent:

1) A is reducible;

2) there exists a regular closed sequence $\{D_\xi : \xi \in [0, \alpha]\}$ such that $A = \bigcup_{\xi \in I} (D_\xi \setminus D_{\xi+1})$ for some $I \subseteq [0, \alpha]$.

Lemma 1. *Let X be a perfectly normal paracompact space and $E \subseteq X$ be a hereditarily Baire subspace. Then E is 1-separated from any G_δ-set $A \subseteq X$ disjoint with E.*

Proof. Fix an arbitrary G_δ-set A such that $A \cap E = \varnothing$ and assume to the contrary that A and E are not 1-separated. Notice that $\overline{A} \cap \overline{E} \neq \varnothing$, otherwise $H = X \setminus \overline{A}$ is F_σ- and G_δ-set which separates A and E.

Let β be the first ordinal of the cardinality greater than $|X|$. We define inductively transfinite sequences of subsets of X by putting $F_0 = X$, $A_0 = A$ and $E_0 = E$. Suppose that for some ordinal number $\alpha < \beta$ there are already constructed sequences $(F_\xi)_{\xi<\alpha}$, $(A_\xi)_{\xi<\alpha}$ and $(E_\xi)_{\xi<\alpha}$ of nonempty subsets of X. We put

$$F_\alpha = \begin{cases} \overline{A_{\alpha-1}} \cap \overline{E_{\alpha-1}}, & \text{if } \alpha \text{ is isolated,} \\ \bigcap\limits_{\xi<\alpha} F_\xi, & \text{if } \alpha \text{ is limit,} \end{cases} \tag{2}$$

$$A_\alpha = A \cap F_\alpha, \qquad E_\alpha = E \cap F_\alpha. \tag{3}$$

We show that the set F_α is nonempty. To obtain a contradiction we suppose that $F_\alpha = \varnothing$. Then sequence

$$X = F_0 \supset \overline{A_0} \supset F_1 \supset \cdots \supset F_\xi \supset \overline{A_\xi} \supset F_{\xi+1} \supset \cdots \supset F_\alpha = \varnothing$$

is regular closed in X. By Proposition 3, the set

$$H = \bigcup_{\xi<\alpha} \left(F_\xi \setminus \overline{A_\xi}\right)$$

is reducible. Moreover, let us check that

$$E \subseteq H \subseteq X \setminus A. \tag{4}$$

Fix $x \in E$ and take $\xi < \alpha$ such that $x \in F_\xi \setminus F_{\xi+1}$. Then $x \in E \cap F_\xi = E_\xi \subseteq \overline{E_\xi}$. Since $x \notin F_{\xi+1}$, $x \notin \overline{A_\xi}$. Hence, $x \in H$.

Now assume $x \in H$ and let $\xi < \alpha$ be such that $x \in F_\xi \setminus \overline{A_\xi}$. If $x \in A$, then $x \in F_\xi \cap A = A_\xi$, a contradiction. Therefore, $x \in X \setminus A$ and (4) is proved. Since X is paracompact, we have that H is F_σ and G_δ in X. By (4), H separates A and E, which implies a contradiction to our assumption. Hence, $F_\alpha \neq \varnothing$.

Therefore, there is a decreasing sequence $(F_\alpha)_{\alpha<\beta}$ of nonempty closed subsets of X and sequences $(A_\alpha)_{\alpha<\beta}$, $(E_\alpha)_{\alpha<\beta}$ of nonempty sets which satisfy (2) and (3) for every $\alpha < \beta$.

We put

$$M = \left\{\xi < \beta : F_\xi \setminus F_{\xi+1} \neq \varnothing\right\} \quad \text{and} \quad N = \left\{\xi < \beta : F_\xi \setminus F_{\xi+1} = \varnothing\right\}.$$

Take $x_\xi \in F_\xi \setminus F_{\xi+1}$ for every $\xi \in M$. Notice that all points x_ξ are distinct. Then

$$|M| = \left|\{x_\xi : \xi \in M\}\right| \leq |X| < |\beta| = |M \cup N|.$$

Hence, $N \neq \varnothing$. Let $\alpha = \min N$. Then $F_\alpha = F_{\alpha+1} = \dots$. Therefore, the equality

$$F_\alpha = \overline{A \cap F_\alpha} \cap \overline{E \cap F_\alpha}$$

is valid by (2) and (3).

Since E is hereditarily Baire and $E \cap F_\alpha$ is a closed subset of E, E_α is a Baire space. Notice that A_α is dense G_δ-subset of F_α. It follows that $F_\alpha \setminus A_\alpha$ is an F_σ-set of the first category in F_α. Hence, E_α as a subset of $F_\alpha \setminus E_\alpha$ is a set of the first category in itself. We obtain a contradiction, because E_α is a Baire space.

Hence, our assumption is not valid and we have that E and A are 1-separated in X. □

4 Extension of Borel 1 functions and infinitely nice sets

Definition 5. *Let X be a topological space. We define $E \subseteq X$ to be (finitely) infinitely nice, if for any disjoint (finite) infinite sequence (E_n) of F_σ- and G_δ-subsets of E such that $E = \bigcup\limits_n E_n$ there exists a disjoint sequence (X_n) of F_σ- and G_δ-subsets of X such that $X = \bigcup\limits_n X_n$ and $X_n \cap E = E_n$ for every n.*

Definition 6. *A subset A of a topological space X is $\mathscr{B}_1$-embedded in X ($\mathscr{B}_1^*$-embedded in X), if every (bounded) Borel 1 function $f : E \to \mathbb{R}$ can be extended to a (bounded) Borel 1 function $g : X \to \mathbb{R}$.*

It was proved in [6, Proposition 8] (see also [8, Theorem 5.3] for functions of the α'th Borel class, $\alpha \geq 1$) that for a perfectly normal space X and a subset $E \subseteq X$ the following properties are equivalent:

(A) E is $\mathscr{B}_1$-embedded in X;

(B) E is 1-separated from any G_δ-set $A \subseteq X$ disjoint with E.

Moreover, it was shown in [8, Theorem 7.2], that property (A) implies

(C) E is infinitely nice.

It is worth noting [8, Proposition 5.1] that the property of E to be finitely nice is equivalent to

(A') E is $\mathscr{B}_1^*$-embedded in X.

Further, it follows from [8, Theorem 7.3] for $\alpha = 1$ that properties (A) and (B) for perfectly normal X are equivalent to the following condition.

(D) For any Polish space Y every Borel 1 function $f : E \to Y$ can be extended to a Borel 1 function $g : X \to Y$.

It is find out that property (C) is equivalent to (A). In order to show this we need to prove the following result.

Proposition 4. *Let X be a perfectly normal space and $E \subseteq X$ be infinitely nice. Then E is $\mathscr{B}_1$-embedded in X.*

Proof. Let $f : E \to \mathbb{R}$ be a Borel 1 function. Without loss of generality, we may assume that $f(E) = \mathbb{R}$.

Fix $n \in \mathbb{N}$. Consider a covering $\{I_{k,n} : k \in \mathbb{Z}\}$ of $\mathbb{R}$ by open intervals

$$I_{k,n} = \left(\frac{k-1}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right).$$

Since f is Borel 1, each set $J_{k,n} = f^{-1}(I_{k,n})$ is F_σ in E and the family $\{J_{k,n} : k \in \mathbb{N}\}$ covers E. By Reduction Theorem [10, §30, VII, Theorem 1] there exists a disjoint family $\{E_{k,n} : k \in \mathbb{N}\}$ of nonempty F_σ- and G_δ-sets in E such that $E_{k,n} \subseteq J_{k,n}$ and $E = \bigcup\limits_k E_{k,n}$. Since E is infinitely nice, there exists a disjoint covering $\{X_{k,n} : k \in \mathbb{N}\}$ of X by F_σ- and G_δ-sets such that $X_{k,n} \cap E = E_{k,n}$.

For every $k, n \in \mathbb{N}$ we pick an arbitrary point $y_{k,n} \in I_{k,n}$. For every $x \in X$ we define

$$f_n(x) = y_{k,n}, \quad \text{if } x \in X_{k,n}.$$

It is not hard to verify that $f_n : X \to \mathbb{R}$ is a Borel 1 function. Notice that for every $x \in E$ and for every $n \in \mathbb{N}$ we have $x \in E_{l,n+1}$ for some integer l. By our construction, there exists $k \in \mathbb{Z}$ such that $E_{l,n+1} \subseteq E_{k,n}$. Hence,

$$|f_{n+1}(x) - f_n(x)| \le \operatorname{diam} I_{k,n} = \frac{1}{2^n}$$

for all $n \in \mathbb{N}$ and $x \in E$. Now for all $x \in X$ we put

$$g_n(x) = \max\Big\{ \min\{f_{n+1}(x) - f_n(x), 2^{-n}\}, -2^{-n}\Big\}.$$

Then $g_n : X \to \mathbb{R}$ is Borel 1. Since $|g_n(x)| \le 2^{-n}$ for all $x \in X$, the series $\sum\limits_{n=1}^{\infty} g_n(x)$ is uniformly convergent on X to a function, say, $g : X \to \mathbb{R}$. Then g is Borel 1 as a sum of uniform convergent series of Borel 1 functions. Moreover, if $x \in E$ and $n \in \mathbb{N}$, then $g_n(x) = f_{n+1}(x) - f_n(x)$ and

$$\sum_{k=1}^{n} g_k(x) = f_{n+1}(x) - f_1(x).$$

Moreover,

$$|f_{n+1}(x) - f(x)| \le \frac{1}{2^n}.$$

Therefore, $f_n \rightrightarrows f$ on E. It remains to put

$$h(x) = g(x) + f_1(x)$$

for every $x \in X$. Hence, h is the required Borel 1 extension of f. □

Now we turn our attention to some examples of $\mathscr{B}_1$-embedded sets which will be useful in the next section.

Proposition 5. *Let X be a perfectly normal space and $E \subseteq X$. If one of the following conditions holds*

(i) E is G_δ;

(ii) E is Lindelöf and hereditarily Baire;

(iii) X is paracompact and E is hereditarily Baire,

then E is $\mathscr{B}_1$-embedded in X.

Proof. In case *(i)*, condition (B) is evident. In case *(ii)*, E satisfies condition (A) according to [6, Theorem 13]. Finally, in case *(iii)*, E satisfies (B) by Lemma 1. □

Remark that in each of cases *(i)*–*(iii)* of Proposition 5 the set E is infinitely nice.

5 Extension of equi-Lebesgue families

Proposition 6. *Let X be a perfectly normal space, E be $\mathscr{B}_1$-embedded in X and let Y be a Polish space. Then every equi-Lebesgue family $\mathscr{F} \subseteq Y^E$ can be extended to an equi-Lebesgue family $\mathscr{G} \subseteq Y^X$.*

Proof. Fix $\varepsilon > 0$ and consider a sequence (E_n) of closed sets in E such that $E = \bigcup_{n=1}^{\infty} E_n$ and $\operatorname{diam} f(E_n) \le \varepsilon$ for every $f \in \mathscr{F}$ and $n \in \mathbb{N}$.

Let $H_1 = E_1$ and $H_n = E_n \setminus \bigcup\limits_{k<n} E_k$. Since E is perfectly normal, then every H_n is F_σ- and G_δ-subset of E. Moreover, (H_n) is mutually disjoint sequence and $E = \bigcup\limits_n H_n$. Since E is infinitely nice, there exists a disjoint sequence (X_n) of F_σ- and G_δ-subsets of X such that $X = \bigcup\limits_n X_n$ and $X_n \cap E = H_n$ for every $n \in \mathbb{N}$.

Take $f \in \mathcal{F}$. Notice that f is Borel 1 since it has Lebesgue property. For every n we fix an arbitrary $y_n^f \in f(H_n)$. Put $g_n^f = f$ on H_n and $g_n = y_n$ on $E \setminus H_n$. It is easy to see that $g_n^f : E \to f(H_n)$ is Borel 1, since H_n is F_σ and G_δ in E. By property (D) there exists a Borel 1 extension $h_n^f : X \to \overline{f(H_n)}$ of g_n^f. Notice that $\operatorname{diam} h_n^f(X) \le \varepsilon$. We put

$$g^f(x) = h_n^f(x),$$

if $x \in X_n$ for some n.

Then $g^f : X \to Y$ is Borel 1 because every X_n is F_σ and G_δ in X. Moreover, $g^f|_E = f$ and $\operatorname{diam} g^f(X_n) \le \varepsilon$ for every n.

It remains to put

$$\mathscr{G} = \left\{ g^f : f \in \mathscr{F} \right\}.$$

□

Remark 2. *Notice that we can not use property (D) for orbit function $f^{\sharp}_{\mathscr{F}} : E \to (Z, \varrho)$, since Z is not separable in general.*

Propositions 5 and 6 imply the following extension theorem.

Theorem 5. *Let X be a perfectly normal space, Y be a Polish space and $E \subseteq X$. If one of the following conditions hold*

(i) E is G_δ;

(ii) E is Lindelöf and hereditarily Baire;

(iii) X is paracompact and E is hereditarily Baire,

then every equi-Lebesgue family $\mathscr{F} \subseteq Y^E$ can be extended to an equi-Lebesgue family $\mathscr{G} \subseteq Y^X$.

Combining Proposition 2 and Theorem 5 *(i)*, we obtain the main result.

Theorem 6. *Let X be a perfectly normal space, Y be a Polish space and $E \subseteq X$ be an arbitrary set. Then every equi-continuous family $\mathscr{F} \subseteq Y^E$ can be extended to equi-Lebesgue family $\mathscr{G} \subseteq Y^X$.*

References

[1] Alikhani-Koopaei A. *Equi-Baire one family of functions on metric spaces: A generalization of equi-continuity; and some applications*. Topology Appl. 2020, **277**, 107170. doi:10.1016/j.topol.2020.107170

[2] Alikhani-Koopaei A. *On dynamics of families of equi-Baire one functions on metric spaces*. Topology Appl. 2022, **322** (3), 108267. doi:10.1016/j.topol.2022.108267

[3] Balcerzak M., Holá L., Holý D. *Properties of equi-Baire 1 and equi-Lebesgue families of functions*. arXiv:2304.07824. doi:10.48550/arXiv.2304.07824

[4] Balcerzak M., Karlova O., Szuca P. *Equi-Baire 1 families of functions*. Topology Appl. 2022, **305**, 107900. doi:10.1016/j.topol.2021.107900

[5] Engelking R. General Topology. Heldermann Verlag, Berlin, 1989.

[6] Kalenda O.F.K., Spurný J. *Extending Baire-one functions on topological spaces*. Topology Appl. 2005, **149**, 195–216. doi:10.1016/J.TOPOL.2004.09.007

[7] Karlova O. *The decomposable and the ambiguous sets*. Carpathian Math. Publ. 2011, **3** (2), 71–76. (in Ukrainian)

[8] Karlova O. *On α-embedded sets and extension of mappings*. Comment. Math. Univ. Carolin. 2013, **54** (3), 377–396.

[9] Koumoullis G. *A generalization of functions of the first class*. Topology Appl. 1993, **50**, 217–239.

[10] Kuratowski K. Topology, vol. 1. Elsevier, Wasrzawa, 1966.

[11] Lebesgue H. *Sur les fonctions représentables analytiqment*. J. Math. Pures Appl. (9) 1905, **1**, 139–216.

[12] Lecomte D. *How can we recover Baire class one functions?* Mathematika 2003, **50** (1–2), 171–198. doi: 10.1112/S0025579300014881

[13] Lee P.Y., Tang W.-K., Zhao D. *An equivalent definition of functions of the first Baire class*. Proc. Amer. Math. Soc. 2001, **129** (8), 2273–2275. doi:10.1090/S0002-9939-00-05826-3

Received 02.11.2024

Карлова О. *Властивість продовження для одностайно лебеґових сімей функцій* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 5–13.

Нехай X — топологічний простір і (Y, d) — повний метричний сепарабельний простір. Сім'ю $\mathscr{F}$ функцій з X в Y ми називаємо одностайно лебеґовою, якщо для кожного $\varepsilon > 0$ існує таке покриття (F_n) простору X, яке складається із замкнених множин, що $\operatorname{diam} f(F_n) \leq \varepsilon$ для всіх $n \in \mathbb{N}$ та $f \in \mathscr{F}$.

Ми доводимо, що для досконало нормального простору X, повного метричного сепарабельного простору Y та довільної підмножини $E \subseteq X$ кожну одностайно неперервну сім'ю функцій $\mathscr{F} \subseteq Y^E$ можна продовжити до одностайно лебеґової сім'ї $\mathscr{G} \subseteq Y^X$.

Ключові слова і фрази: продовження функцій першого класу Бореля, одностайно берівська сім'я функцій, одонстайно лебеґова сім'я функцій, 1-відокремна множина, метризовний простір, топологічний простір.