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Best m-term trigonometric approximations of the isotropic Nikol'skii-Besov-type classes of periodic functions of several variables

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We obtained the exact order estimates of the best m-term trigonometric approximations of the isotropic Nikol'skii-Besov-type classes $B^{\omega}_{p,\theta}$ of periodic functions of several variables in the spaces $B_{q,1}$ for $1 < p < q < \infty$, $q \geq 2$. A peculiarity of these spaces, as linear subspaces of L_q, is that the norm in them is stronger than the L_q-norm. It was found that the obtained estimates of the considered approximation characteristic coincide in order with the estimates of the corresponding characteristic of the classes $B^{\omega}_{p,\theta}$ in the spaces L_q.

Key words and phrases: periodic function of several variables, Nikol'skii-Besov-type class, best m-term trigonometric approximation.

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Introduction

In the paper, we investigate the best m-term trigonometric approximations of the isotropic Nikol'skii-Besov-type classes $B^{\omega}_{p,\theta}$ of periodic functions of several variables in the spaces $B_{q,1}$ for $1 < p < q < \infty$, $q \geq 2$. The norm in these spaces is stronger than the L_q-norm.

In the number of papers [5, 10–12, 17, 18, 20, 24, 32–35, 38, 43], the questions were investigated concerning approximation of classes of periodic functions of several variables with mixed smoothness, namely, the classes of Nikol'skii-Besov-type $B^{r}_{p,\theta}$, Sobolev classes $W^{r}_{p,\alpha}$ and some their analogs, in the spaces with slightly modified norms comparing to the norm of the spaces $B_{q,1}$, which we consider.

It is important to note that in the mentioned works, in almost all situations, there were found differences in the order estimates of approximation characteristics in the spaces L_q compared to the spaces $B_{q,1}$.

An essentially different situation is observed in the study of the isotropic Nikol'skii-Besov-type classes $B^{\omega}_{p,\theta}$ in the spaces $B_{q,1}$, $1 \leq q \leq \infty$. According to the results of our research on the best m-term trigonometric approximations of the classes $B^{\omega}_{p,\theta}$, it turned out that their estimates in the spaces $B_{q,1}$ and L_q have the same orders. Note that a similar circumstance was noted in the work [15] when studying other approximation characteristics of the classes $B^{\omega}_{p,\theta}$.

The work consists of three parts. The first part plays an auxiliary role. Here we introduce necessary notation and define the classes of functions $B^{\omega}_{p,\theta}$ and spaces $B_{q,1}$, in which we es-

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timate the approximation error. In the second part we obtain estimates of the best m-term trigonometric approximations of the classes $B^{\omega}_{p,\theta}$ in the space $B_{q,1}$ for $1 < p < q < \infty$, $q \geq 2$. The third part of the work is devoted to comments on the obtained statements. The main results are contained in Theorems 1–3. They complement and generalize the corresponding statements obtained in the works [7,13,27,37,46].

1 Functional classes $B^{\Omega}_{p,\theta}$ and spaces $B_{q,1}$

Let $\mathbb{R}^d$ be a d-dimensional space with the elements $x = (x_1, \dots, x_d)$. For any $x, y \in \mathbb{R}^d$, let $(x, y) = x_1 y_1 + \dots + x_d y_d$ be their scalar product.

By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d := \prod_{j=1}^{d} [0, 2\pi)$, we denote the space of 2π-periodic in each variable functions f, for which

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} = \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

For $f \in L_p(\mathbb{T}^d)$, we denote $\Delta_h f(x) := f(x+h) - f(x)$, $h \in \mathbb{R}^d$. Then the multiple difference of the order $l \in \mathbb{N}$ of the function $f(x)$ at the point $x = (x_1, \dots, x_d)$ with the step $h = (h_1, \dots, h_d)$ we define by the formula

$$\Delta^l_h f(x) := \Delta_h \Delta^{l-1}_h f(x), \quad \Delta^0_h f(x) := f(x).$$

On the basis of the multiple difference $\Delta^l_h f(x)$, let us determine the module of continuity of the lth order of the function $f \in L_p(\mathbb{T}^d)$ using the formula

$$\omega_l(f, t) := \sup_{|h| \leq t} \|\Delta^l_h f(\cdot)\|_p, \quad \text{where} \quad |h| = \sqrt{h_1^2 + \dots + h_d^2}.$$

Let $\omega(t)$ be a function of the type of modulus of continuity of the lth order, i.e. $\omega(t)$ satisfies the following conditions at $t \geq 0$:

1) $\omega(0) = 0$; $\omega(t) > 0$ at $t > 0$;

2) $\omega(t)$ is continuous;

3) $\omega(t)$ increases;

4) for every $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\omega(nt) \leq C_1 n^l \omega(t)$, where the constant $C_1 > 0$ does not depend on n and t.

We also assume that $\omega(t)$ satisfies conditions (S^α) and (S_l), which are called the Bari-Stechkin conditions [2]. This means the following.

A function $\omega(\tau) \geq 0$ satisfies the condition (S^α) if $\frac{\omega(\tau)}{\tau^\alpha}$ almost increases for some $\alpha > 0$, i.e. there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\omega(\tau_1)}{\tau_1^\alpha} \leq C_2 \frac{\omega(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \leq \tau_2.$$

A function $\omega(\tau) \geq 0$ satisfies the condition (S_l) if $\frac{\omega(\tau)}{\tau^\gamma}$ almost decreases for some $0 < \gamma < l$, i.e. there exists a constant $C_3 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\omega(\tau_1)}{\tau_1^\gamma} \geq C_3 \frac{\omega(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \leq \tau_2.$$

We say that a function $f \in L_p(\mathbb{T}^d)$ belongs to the space $B^\omega_{p,\theta}$, $1 \leq p, \theta \leq \infty$ (see, e.g., [13]), if

$$\left(\int_0^\infty \left(\frac{\omega_l(f,t)_p}{\omega(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}} < \infty, \quad 1 \leq \theta < \infty; \quad \sup_{t>0} \frac{\omega_l(f,t)_p}{\omega(t)} < \infty, \quad \theta = \infty,$$

where $\omega(t)$ is a function of the type of modulus of continuity of the lth order.

The norm in the space $B^\omega_{p,\theta}$ is defined by the formula

$$\|f\|_{B^\omega_{p,\theta}} := \begin{cases} \|f\|_p + \left(\int_0^\infty \left(\frac{\omega_l(f,t)_p}{\omega(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}}, & 1 \leq \theta < \infty, \\ \|f\|_p + \sup\limits_{t>0} \frac{\omega_l(f,t)_p}{\omega(t)}, & \theta = \infty. \end{cases}$$

If $\omega(t) = t^r$, $0 < r < l$, then the spaces $B^\omega_{p,\theta}$ coincide with the Besov spaces $B^r_{p,\theta}$ [6]; in particular, $B^r_{p,\infty} \equiv H^r_p$ at $\theta = \infty$, where H^r_p are spaces introduced by S.M. Nikol'skii [22].

Note that as the parameter θ increases, the spaces $B^\omega_{p,\theta}$ expand, i.e. the following embeddings are valid for $1 \leq \theta_1 \leq \theta_2 \leq \infty$:

$$B^\omega_{p,1} \subset B^\omega_{p,\theta_1} \subset B^\omega_{p,\theta_2} \subset B^\omega_{p,\infty} \equiv H^\omega_p.$$

In what follows, it will be convenient to use the equivalent (with accuracy to absolute constants) definition of the norm in the space $B^\omega_{p,\theta}$.

For $f \in L_p(\mathbb{T}^d)$, $1 < p < \infty$, let us introduce the notation

$$f_{(0)} := f_{(0)}(x) = \widehat{f}(0), \quad f_{(s)} := f_{(s)}(x) = \sum_{k \in \mu(s)} \widehat{f}(k)\, e^{i(k,x)}, \quad s \in \mathbb{N},$$

where $\mu(s) = \left\{ k \in \mathbb{Z}^d : 2^{s-1} \leq \max_{j=1,\dots,d} |k_j| < 2^s \right\}$, $(k,x) = k_1x_1 + \cdots + k_dx_d$, and

$$\widehat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f.

Then, for $1 < p < \infty$ and under the conditions 1)–4), (S^α) with some $\alpha > 0$ and (S_l), the following relations hold (see, e.g., [13]):

$$\begin{aligned} \|f\|_{B^\omega_{p,\theta}} &\asymp \left(\sum_{s=0}^\infty \omega^{-\theta}(2^{-s})\, \|f_{(s)}\|_p^\theta \right)^{\frac{1}{\theta}}, \quad 1 \leq \theta < \infty, \\ \|f\|_{B^\omega_{p,\infty}} &\asymp \sup_{s \in \mathbb{Z}_+} \frac{\|f_{(s)}\|_p}{\omega(2^{-s})}, \quad \theta = \infty. \end{aligned} \tag{1}$$

Hereafter, for positive sequences $a(n)$ and $b(n)$, $n \in \mathbb{N}$, we use the notation $a(n) \asymp b(n)$, which means that there exist the constants $0 < C_4 < C_5$ such that $C_4 a(n) \leq b(n) \leq C_5 a(n)$. If only $b(n) \leq C_5 a(n)$ $\big($or $b(n) \geq C_4 a(n)\big)$ is satisfied, then we write $b(n) \ll a(n)$ (or $b(n) \gg a(n)$).

Below, we will consider classes of functions $\mathbf{B}^{\omega}_{p,\theta}$ from the spaces $B^{\omega}_{p,\theta}$ that are defined by

$$\mathbf{B}^{\omega}_{p,\theta} := \big\{ f \in B^{\omega}_{p,\theta} : \|f\|_{B^{\omega}_{p,\theta}} \leq 1 \big\}.$$

Note that from the viewpoint of their approximation characteristics, the classes $\mathbf{B}^{\omega}_{p,\theta}$ were studied in a number of works (see, e.g., [13,15,36,40,46,47] and the references therein).

Now let us define the norm $\|\cdot\|_{B_{q,1}}$ of functions $f \in L_q(\mathbb{T}^d)$ in the spaces $B_{q,1}$, $1 < q < \infty$, which is similar to the decomposition norm of functions from the spaces of the Nikol'skii-Besov-type $B^{\omega}_{p,\theta}$ (see (1)).

We also note that for trigonometric polynomial P according to the multiple trigonometric system $\{e^{i(k,x)}\}_{k \in \mathbb{Z}^d}$, the norm $\|P\|_{B_{q,1}}$, $1 < q < \infty$, is defined by the formula

$$\|P\|_{B_{q,1}} := \sum_{s} \|P_{(s)}\|_q.$$

Similarly we define the norm $\|f\|_{B_{q,1}}$, $1 < q < \infty$, for any function $f \in L_q(\mathbb{T}^d)$, such that the series $\sum_{s \in \mathbb{Z}_+} \|f_{(s)}\|_q$ is convergent. Note that for $f \in B_{q,1}$, $1 < q < \infty$, the following relation

$$\|f\|_q \ll \|f\|_{B_{q,1}}$$

holds.

2 Best m-term trigonometric approximations

Let us define the approximation characteristic that will be investigated in this part of the paper.

Let X be a normed space with the norm $\|\cdot\|_X$ and Θ_m be a set of m arbitrary d-dimensional vectors with integer coordinates. Let us denote by

$$P(\Theta_m) := P(\Theta_m, x) = \sum_{k \in \Theta_m} c_k e^{i(k,x)}, \quad c_k \in \mathbb{C},$$

a trigonometric polynomial with "numbers" of harmonic from the set Θ_m. For $f \in X$, we consider the quantity

$$e_m(f)_X := \inf_{c_k} \inf_{\Theta_m} \|f - P(\Theta_m)\|_X,$$

which is called the best m-term trigonometric approximation of f. For a class $F \subset X$, we set

$$e_m(F)_X := \sup_{f \in F} e_m(f)_X.$$

The quantity $e_m(f)_2 := e_m(f)_{L_2(\mathbb{T})}$ for univariate functions was introduced by S.B. Stechkin [42] in order to formulate a criterion of absolute convergence of orthogonal series in the general case of approximations by polynomials with respect to arbitrary orthogonal system in a Hilbert space.

Further the quantities $e_m(F)_X$ for certain functional classes and spaces $X = L_q(\mathbb{T}^d)$, $d \geq 1$, as well as other normed spaces, were studied by many authors. The detailed overview can be found in the papers [1,3–5,7,14,20,23,28,30,31,37,39,41,46] and monographs [8,26,44,45].

Before proceeding directly to the obtained results, let us formulate the well-known statements that we will use.

Lemma A ([4]). *Let $2 < q < \infty$. Then for any trigonometric polynomial $P(\Theta_n, x)$ and any $m < n$, one can find a trigonometric polynomial $P(\Theta_m, x)$ such that*

$$\|P(\Theta_n) - P(\Theta_m)\|_q \le C_6(q)\sqrt{\frac{n}{m}}\|P(\Theta_n)\|_2,$$

and, moreover, $\Theta_m \subset \Theta_n$.

Theorem A. *Let*

$$t(x) = \sum_{|k_j| \le n_j} c_k e^{i(k,x)},$$

where $n_j \in \mathbb{N}, j = 1, \ldots, d$. Then for $1 \le p < q \le \infty$ the following inequality

$$\|t\|_q \le 2^d \prod_{j=1}^{d} n_j^{\frac{1}{p}-\frac{1}{q}} \|t\|_p$$

holds.

The above inequality was obtained by S.M. Nikol'skii [22] and is referred to as the inequality for different metrics.

Theorem B ([46]). *Let $1 < p < q < \infty, q \ge 2, 1 \le \theta \le \infty$, and $\omega(\tau)$ satisfies the condition (S^α) with $\alpha > \max\{\frac{d}{p}; \frac{d}{2}\}$ and the condition (S_l). Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_q \asymp \omega(m^{-\frac{1}{d}}) m^{\left(\frac{1}{p} - \max\left\{\frac{1}{q}; \frac{1}{2}\right\}\right)_+}$$

holds, where $a_+ = \max\{a; 0\}$.

Next, we proceed to the formulation and proving of the obtained results.

Theorem 1. *Let $1 < p \le 2 < q < \infty, 1 \le \theta \le \infty$, and $\omega(t)$ satisfies the condition (S^α) with some $\alpha > d\left(\frac{1}{p} - \frac{1}{q}\right)$ and the condition $(S_{\min\{d/p; l\}})$. Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp \omega(m^{-\frac{q}{2d}}) m^{\frac{q}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \tag{2}$$

is valid.

Proof. First, we prove the upper estimate. The right-hand side in (2) does not depend on the parameter θ. We also note, that due to the embedding $\mathbf{B}^{\omega}_{p,\theta} \subset \mathbf{B}^{\omega}_{p,\infty} \equiv \mathbf{H}^{\omega}_p$, $1 \le \theta < \infty$, it is sufficient to get the required estimate for the quantity $e_m(\mathbf{H}^{\omega}_p)_{B_{q,1}}$.

So, let $n \in \mathbb{N}$ be such that $2^{dn} < m \le 2^{d(n+1)}$ and $f \in \mathbf{H}^{\omega}_p$. Then we can write

$$f(x) = \sum_{s=0}^{\infty} f_{(s)}(x)$$

and

$$\|f_{(s)}\|_p \ll \omega(2^{-s}). \tag{3}$$

We will approximate the function $f \in \mathbf{H}^{\omega}_p$ by the polynomials $P(\Theta_m)$ of the form

$$P(\Theta_m) := P(\Theta_m, x) := \sum_{s=0}^{n-1} f_{(s)}(x) + \sum_{n \le s < \beta n} P(\Theta_{m_s}, x), \tag{4}$$

where $P(\Theta_{m_s}, x)$ are the polynomials that approximate the "blocks" $f_{(s)}(x)$ according to Lemma A. We choose the numbers β and m_s as follows

$$\beta = \frac{q}{2}; \quad m_s := \Big[2^{dn} 2^{-\frac{q}{2}dn} \omega^{-1}\big(2^{-\frac{qn}{2}}\big) 2^{\frac{ds}{p}} \omega(2^{-s})\Big] + 1, \tag{5}$$

where $[a]$ denotes the integer part of the number a.

Let us show that with this choice of the numbers β and m_s, the polynomial $P(\Theta_m)$ contains no more than m harmonics in order.

By $|\mu(s)|$ we denote the number of elements of the set $\mu(s)$. Then we can write

$$\sum_{s=0}^{n-1} |\mu(s)| + \sum_{n \le s < \frac{qn}{2}} m_s \ll 2^{dn} + \Big(\frac{q}{2} - 1\Big) n + 2^{dn} 2^{-\frac{dqn}{2p}} \omega^{-1}\big(2^{-\frac{qn}{2}}\big) \sum_{n \le s < \frac{qn}{2}} 2^{\frac{ds}{p}} \omega(2^{-s}) = I_1. \tag{6}$$

Further, since $\omega(t)$ satisfies the condition $(S_{\min\{d/p; l\}})$, then the relation

$$\frac{\omega(2^{-s})}{2^{-\gamma s}} \ll \frac{\omega\big(2^{-\frac{qn}{2}}\big)}{2^{-\frac{\gamma qn}{2}}}, \quad 0 \le s < \frac{qn}{2}, \quad 0 < \gamma < \min\Big\{\frac{d}{p}; l\Big\}$$

holds. Taking it into account, the estimation of the quantity I_1 can be continued as follows

$$\begin{aligned} I_1 &\ll 2^{dn} + 2^{dn} 2^{-\frac{dqn}{2p}} \omega^{-1}\big(2^{-\frac{qn}{2}}\big) \sum_{n \le s < \frac{qn}{2}} \frac{\omega(2^{-s})}{2^{-\gamma s}} 2^{s\left(\frac{d}{p} - \gamma\right)} \\ &\ll 2^{dn} + 2^{dn} 2^{-\frac{dqn}{2p}} \omega^{-1}\big(2^{-\frac{qn}{2}}\big) \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{\gamma qn}{2}} 2^{\frac{qn}{2}\left(\frac{d}{p} - \gamma\right)} \ll 2^{dn} \asymp m. \end{aligned} \tag{7}$$

Therefore, comparing (6) and (7), we conclude that the number of harmonics of the polynomial $P(\Theta_m)$ does not exceed m in order.

Thus, according to the choice of the polynomial $P(\Theta_m)$ and taking into account the property of the norm $\|\cdot\|_{B_{q,1}}$, we can write

$$\|f - P(\Theta_m)\|_{B_{q,1}} \ll \Big\| \sum_{n \le s < \frac{qn}{2}} \big(f_{(s)} - P(\Theta_{m_s})\big) \Big\|_{B_{q,1}} + \Big\| \sum_{s \ge \frac{qn}{2}} f_{(s)} \Big\|_{B_{q,1}} = I_2 + I_3. \tag{8}$$

Let us first estimate the quantity I_3. Taking into account the definition of the norm $\|\cdot\|_{B_{q,1}}$, by the inequality of different metrics (Theorem A) and the estimate (3), we get

$$I_3 = \Big\| \sum_{s \ge \frac{qn}{2}} f_{(s)} \Big\|_{B_{q,1}} \asymp \sum_{s \ge \frac{qn}{2}} \|f_{(s)}\|_q \ll \sum_{s \ge \frac{qn}{2}} 2^{ds\left(\frac{1}{p} - \frac{1}{q}\right)} \|f_{(s)}\|_p \ll \sum_{s \ge \frac{qn}{2}} 2^{ds\left(\frac{1}{p} - \frac{1}{q}\right)} \omega(2^{-s}). \tag{9}$$

Since $\omega(t)$ satisfies the condition (S^{α}) with $\alpha > d\big(\frac{1}{p} - \frac{1}{q}\big)$, then the relation

$$\frac{\omega(2^{-s})}{2^{-\alpha s}} \ll \frac{\omega\big(2^{-\frac{qn}{2}}\big)}{2^{-\frac{\alpha qn}{2}}}, \quad s \ge \frac{qn}{2},$$

holds, and therefore the estimate (9) takes the form

$$\begin{aligned} I_3 &\ll \sum_{s \ge \frac{qn}{2}} 2^{ds\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\omega(2^{-s})}{2^{-\alpha s}} 2^{-\alpha s} \ll \frac{\omega\big(2^{-\frac{qn}{2}}\big)}{2^{-\frac{\alpha qn}{2}}} \sum_{s \ge \frac{qn}{2}} 2^{-s\left(\alpha - d\left(\frac{1}{p} - \frac{1}{q}\right)\right)} \\ &\ll \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \asymp \omega\big(2^{-\frac{q}{2d}}\big) m^{\frac{q}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}. \end{aligned} \tag{10}$$

Moving to estimation of the term I_2, we use Lemma A, Theorem A, and estimate (3). Hence, we get

$$I_2 = \Big\| \sum_{n\le s<\frac{qn}{2}} (f_{(s)} - P(\Theta_{m_s})) \Big\|_{B_{q,1}} \asymp \sum_{n\le s<\frac{qn}{2}} \|(f_{(s)} - P(\Theta_{m_s}))\|_q \ll \sum_{n\le s<\frac{qn}{2}} \Big(\frac{2^{ds}}{m_s}\Big)^{\frac12} \|f_{(s)}\|_2$$
$$\ll \sum_{n\le s<\frac{qn}{2}} \frac{2^{\frac{ds}{2}} 2^{ds\left(\frac1p-\frac12\right)} \|f_{(s)}\|_p}{m_s^{\frac12}} = \sum_{n\le s<\frac{qn}{2}} \frac{2^{\frac{ds}{p}} \|f_{(s)}\|_p}{m_s^{\frac12}} \ll \sum_{n\le s<\frac{qn}{2}} \frac{2^{\frac{ds}{p}} \omega(2^{-s})}{m_s^{\frac12}}. \quad (11)$$

Substituting the values of m_s from (5) into (11), we obtain the estimate of the quantity I_2:

$$\begin{aligned} I_2 &\ll \sum_{n\le s<\frac{qn}{2}} 2^{\frac{ds}{p}} \omega(2^{-s}) 2^{-\frac{dn}{2}} 2^{\frac{dqn}{4p}} \omega^{\frac12}\big(2^{-\frac{qn}{2}}\big) 2^{-\frac{ds}{2p}} \omega^{-\frac12}(2^{-s}) \\ &\ll 2^{-\frac{dn}{2}} 2^{\frac{dqn}{4p}} \omega^{\frac12}\big(2^{-\frac{qn}{2}}\big) \sum_{n\le s<\frac{qn}{2}} 2^{\frac{ds}{2p}} \omega^{\frac12}(2^{-s}) \\ &= 2^{-\frac{dn}{2}} 2^{\frac{dqn}{4p}} \omega^{\frac12}\big(2^{-\frac{qn}{2}}\big) \sum_{n\le s<\frac{qn}{2}} \frac{\omega^{\frac12}(2^{-s})}{2^{-\frac{\gamma s}{2}}} 2^{s\left(\frac{d}{2p}-\frac{\gamma}{2}\right)} \\ &\ll 2^{-\frac{dn}{2}} 2^{\frac{dqn}{4p}} \omega^{\frac12}\big(2^{-\frac{qn}{2}}\big) \frac{\omega^{\frac12}\big(2^{-\frac{qn}{2}}\big)}{2^{-\frac{\gamma qn}{2}}} \sum_{n\le s<\frac{qn}{2}} 2^{\frac{s}{2}\left(\frac{d}{p}-\gamma\right)} \\ &\ll 2^{-\frac{dn}{2}} 2^{\frac{dqn}{4p}} \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{\gamma qn}{2}} 2^{\frac{dqn}{4p}} 2^{-\frac{\gamma qn}{2}} \\ &= 2^{-\frac{dn}{2}} 2^{\frac{dqn}{2p}} \omega\big(2^{-\frac{qn}{2}}\big) = 2^{\frac{dqn}{2}\left(\frac1p-\frac1q\right)} \omega\big(2^{-\frac{qn}{2}}\big) \\ &\asymp \omega\big(m^{-\frac{q}{2d}}\big) m^{\frac{q}{2}\left(\frac1p-\frac1q\right)}. \end{aligned} \quad (12)$$

Combining (12), (10) and using (8), we obtain the required upper estimate for the quantity $e_m(\mathbf{H}^\omega_p)_{B_{q,1}}$, and hence for $e_m(\mathbf{B}^\omega_{p,\theta})_{B_{q,1}}$, $1 \le \theta < \infty$.

Let us get the respective lower estimate in (2). Note that due to the embedding $\mathbf{B}^\omega_{p,1} \subset \mathbf{B}^\omega_{p,\theta}$, $1 < \theta \le \infty$, it suffices to obtain it for the classes $\mathbf{B}^\omega_{p,1}$.

We will use for $f \in \mathbf{B}_{q,1}$ the relation following from a more general result of S.M. Nikol'skii (see, e.g., [21, p. 25]). In our notation it takes the form

$$e_m(f)_{B_{q,1}} \gg e_m(f)_q = \inf_{\Theta_m} \sup_{\substack{P\in L^\perp(\Theta_m)\\ \|P\|_{q'}\le 1}} \left| \int_{\mathbb{T}^d} f(x)P(x)\,dx \right|, \quad (13)$$

where $\frac1q + \frac1{q'} = 1$, and $L^\perp(\Theta_m)$ denotes the set of functions that are orthogonal to the subspace of trigonometric polynomials with "numbers" of harmonics from the set Θ_m.

First, we construct a function $P_1(x)$ that satisfies the conditions (13) for the function $P(x)$.

Let the numbers $m, n \in \mathbb{N}$ be such that $2^{dn} < m \le 2^{d(n+1)}$, i.e. the relation $m \asymp 2^{dn}$ holds. Consider the function

$$g(x) = \sum_{|k_j|<2^{\frac{qn}{2}}} e^{i(k,x)},$$

and put

$$g_1(x) = g(x) - \sum_{k\in\Theta_m}{}^{*} e^{i(k,x)},$$

where $\sum_{k\in\Theta_m}{}^{*} e^{i(k,x)}$ is a polynomial that contains only those harmonics $e^{i(k,x)}$ of the function $g(x)$ that have "numbers" from the set Θ_m.

Further we use the known relation (see, e.g., [19])

$$\Bigg\| \sum_{\substack{|k_j|<2^l \\ j=1,\dots,d}} e^{i(k,\cdot)} \Bigg\|_q \asymp 2^{dl\left(1-\frac{1}{q}\right)}, \quad 1<q<\infty. \tag{14}$$

So, taking into account that $1<q'\le 2$ and using (14), we can write

$$\|g_1\|_{q'} \le \|g\|_{q'} + \Bigg\| \sum_{k\in\Theta_m}{}^{*} e^{i(k,\cdot)} \Bigg\|_2 \ll 2^{\frac{dqn}{2}\left(1-\frac{1}{q'}\right)} + \sqrt{m} \asymp 2^{\frac{dn}{2}} + 2^{\frac{dn}{2}} \asymp 2^{\frac{dn}{2}}.$$

From this we conclude that the function $P_1(x) = C_7 2^{-\frac{dn}{2}} g_1(x)$, with the corresponding constant $C_7>0$, satisfies the conditions of the relation (13).

Now consider the function

$$f_1(x) = C_8 \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} g(x), \quad C_8>0,$$

and show that it belongs to the class $\mathbf{B}^{\omega}_{p,1}$ with certain constant $C_8>0$.

Using the estimate (14) and taking into account that $\omega(t)$ satisfies the condition (S^{α}), we have

$$\begin{aligned}
\|f_1\|_{B^{\omega}_{p,1}} &\asymp \sum_{s=0}^{\infty} \omega^{-1}(2^{-s}) \|(f_1)_{(s)}\|_p \\
&\ll \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} \sum_{0\le s<\frac{qn}{2}} \omega^{-1}(2^{-s}) \|g_{(s)}\|_p \\
&\asymp \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} \sum_{0\le s<\frac{qn}{2}} \omega^{-1}(2^{-s}) 2^{ds\left(1-\frac{1}{p}\right)} \\
&\ll \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} 2^{\frac{dqn}{2}\left(1-\frac{1}{p}\right)} \sum_{0\le s<\frac{qn}{2}} \frac{\omega^{-1}(2^{-s})}{2^{\alpha s}} 2^{\alpha s} \\
&\ll \omega\big(2^{-\frac{qn}{2}}\big) \omega^{-1}\big(2^{-\frac{qn}{2}}\big) 2^{-\frac{\alpha qn}{2}} \sum_{0\le s<\frac{qn}{2}} 2^{\alpha s} \\
&\ll \omega\big(2^{-\frac{qn}{2}}\big) \omega^{-1}\big(2^{-\frac{qn}{2}}\big) 2^{-\frac{\alpha qn}{2}} 2^{\frac{\alpha qn}{2}} = 1.
\end{aligned}$$

So, we conclude that $f_1 \in \mathbf{B}^{\omega}_{p,1}$ with the corresponding constant $C_8>0$.

Thus, using the relation (13) for the functions $f_1(x)$ and $P_1(x)$, we obtain

$$\begin{aligned}
e_m(f_1)_{B_{q,1}} &\gg \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} 2^{-\frac{dn}{2}} \big(\|g\|_2^2 - m\big) \\
&\asymp \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-1\right)} 2^{-\frac{dn}{2}} 2^{\frac{dqn}{2}} \\
&= \omega\big(2^{-\frac{qn}{2}}\big) 2^{\frac{dqn}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \\
&\asymp \omega\big(m^{-\frac{q}{2d}}\big) m^{\frac{q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}$$

$\square$

Let us formulate a consequence from the obtained result, which concerns the quantity $e_m(\mathbf{B}^{\omega}_{p,\theta})_q := e_m(\mathbf{B}^{\omega}_{p,\theta})_{L_q(\mathbb{T}^d)}$.

Corollary 1. *Let $1 < p \le 2 < q < \infty$, $1 \le \theta \le \infty$, and $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > d\big(\frac{1}{p} - \frac{1}{q}\big)$ and the condition $(S_{\min\{d/p;l\}})$. Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_q \asymp \omega\big(m^{-\frac{q}{2d}}\big) m^{\frac{q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \tag{15}$$

holds.

Proof. The upper estimate in (15) follows from Theorem 1 and the relation $\|\cdot\|_q \le \|\cdot\|_{B_{q,1}}$. The corresponding lower estimate was also established in the proof of Theorem 1. $\square$

Remark 1. *If $\omega(t) = t^r, r > 0$, then, as noted above, $\mathbf{B}^{\omega}_{p,\theta} = \mathbf{B}^{r}_{p,\theta}$, and therefore, under the condition $d\big(\frac{1}{p} - \frac{1}{q}\big) < r < \frac{d}{p}$, $1 < p \le 2 < q < \infty$, from* (2) *we obtain*

$$e_m(\mathbf{B}^{r}_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{q}{2}\left(\frac{r}{d}-\frac{1}{p}+\frac{1}{q}\right)}.$$

The above estimate was established in the paper [23].

In the following statement, we establish the order of quantity $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$ for other values of the smoothness parameter α and, accordingly, other behavior of the function $\omega(t)$.

Theorem 2. *Let $1 < p \le 2 < q < \infty$, $1 \le \theta \le \infty$, and $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > \frac{d}{p}$ and the condition (S_l). Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp \omega\big(m^{-\frac{1}{d}}\big) m^{\frac{1}{p}-\frac{1}{2}} \tag{16}$$

is valid.

Proof. First, we obtain the upper estimate in (16) for the quantity $e_m(\mathbf{H}^{\omega}_{p})_{B_{q,1}}$. To approximate the function $f \in \mathbf{H}^{\omega}_{p}$ we use the polynomial $P(\Theta_m)$ of the form (4). In this case, we choose the numbers β and m_s as follows

$$\beta = \frac{\frac{\alpha}{d} - \frac{1}{p} + \frac{1}{2}}{\frac{\alpha}{d} - \frac{1}{p} + \frac{1}{q}}, \qquad m_s := \Big[\omega^{-1}(2^{-n})2^{\frac{dn}{p'}}\omega(2^{-s})2^{\frac{ds}{p}}\Big] + 1, \tag{17}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Then, taking into account that $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > \frac{d}{p}$, for the number of harmonics of the polynomial $P(\Theta_m)$ we obtain the estimate

$$\begin{aligned}
\sum_{s=0}^{n-1} |\mu(s)| + \sum_{n \le s < \beta n} m_s &\ll 2^{dn} + (\beta-1)n + \omega^{-1}(2^{-n})2^{\frac{dn}{p'}} \sum_{n \le s < \beta n} \omega(2^{-s})2^{\frac{ds}{p}} \\
&\ll 2^{dn} + (\beta-1)n + \omega^{-1}(2^{-n})2^{\frac{dn}{p'}} \sum_{n \le s < \beta n} \frac{\omega(2^{-s})}{2^{-\alpha s}} 2^{-s\left(\alpha - \frac{d}{p}\right)} \\
&\ll 2^{dn} + (\beta-1)n + \omega^{-1}(2^{-n})2^{\frac{dn}{p'}}\omega(2^{-n})2^{\alpha n} \sum_{n \le s < \beta n} 2^{-s\left(\alpha - \frac{d}{p}\right)} \\
&\ll 2^{dn} + (\beta-1)n + 2^{\frac{dn}{p'}}2^{\frac{dn}{p}} \ll 2^{dn} \asymp m.
\end{aligned}$$

Further, by analogy with (8) for $f \in \mathbf{H}^{\omega}_p$ we write

$$\|f - P(\Theta_m)\|_{B_{q,1}} \ll \Big\| \sum_{n \le s < \beta n} (f_{(s)} - P(\Theta_{m_s})) \Big\|_{B_{q,1}} + \Big\| \sum_{s \ge \beta n} f_{(s)} \Big\|_{B_{q,1}} = I_4 + I_5, \tag{18}$$

where the numbers β and m_s satisfy (17).

Let us first estimate the term I_4. We repeat the same considerations that were used to establish the estimate of the quantity I_2. Taking into account the values of β and m_s, as well as the condition $\alpha > \frac{d}{p}$, we obtain

$$\begin{aligned} I_4 &\ll \sum_{n \le s < \beta n} \frac{2^{\frac{ds}{p}} \omega(2^{-s})}{m_s^{\frac{1}{2}}} \ll \omega^{\frac{1}{2}}(2^{-n}) 2^{-\frac{dn}{2p'}} \sum_{n \le s < \beta n} \omega^{\frac{1}{2}}(2^{-s}) 2^{\frac{ds}{2p}} \\ &= \omega^{\frac{1}{2}}(2^{-n}) 2^{-\frac{dn}{2p'}} \sum_{n \le s < \beta n} \frac{\omega^{\frac{1}{2}}(2^{-s})}{2^{-\frac{\alpha s}{2}}} 2^{-\frac{s}{2}\left(\alpha - \frac{d}{p}\right)} \\ &\ll \omega^{\frac{1}{2}}(2^{-n}) 2^{-\frac{dn}{2p'}} \omega^{\frac{1}{2}}(2^{-n}) 2^{\frac{\alpha n}{2}} \sum_{n \le s < \beta n} 2^{-\frac{s}{2}\left(\alpha - \frac{d}{p}\right)} \\ &\ll \omega(2^{-n}) 2^{\frac{dn}{p}} 2^{-\frac{dn}{2}} = \omega(2^{-n}) 2^{dn\left(\frac{1}{p} - \frac{1}{2}\right)} \asymp \omega(m^{-\frac{1}{d}}) m^{\frac{1}{p} - \frac{1}{2}}. \end{aligned} \tag{19}$$

Similarly to (9), taking into account the value of β and the condition $\alpha > \frac{d}{p}$ we obtain the following estimate of the quantity I_5

$$\begin{aligned} I_5 &\ll \sum_{s \ge \beta n} 2^{ds\left(\frac{1}{p} - \frac{1}{q}\right)} \omega(2^{-s}) = \sum_{s \ge \beta n} 2^{ds\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\omega(2^{-s})}{2^{-\alpha s}} 2^{-\alpha s} \\ &\ll \frac{\omega(2^{-\beta n})}{2^{-\alpha \beta n}} \sum_{s \ge \beta n} 2^{-s\left(\alpha - d\left(\frac{1}{p} - \frac{1}{q}\right)\right)} \ll \omega(2^{-\beta n}) 2^{\beta n d\left(\frac{1}{p} - \frac{1}{q}\right)} \\ &\ll \frac{\omega(2^{-\beta n})}{2^{-\alpha \beta n}} 2^{-\beta n\left(\alpha - d\left(\frac{1}{p} - \frac{1}{q}\right)\right)} \ll \frac{\omega(2^{-n})}{2^{-\alpha n}} 2^{-n\left(\alpha - d\left(\frac{1}{p} - \frac{1}{2}\right)\right)} \\ &= \omega(2^{-n}) 2^{dn\left(\frac{1}{p} - \frac{1}{2}\right)} \asymp \omega\big(m^{-\frac{1}{d}}\big) m^{\frac{1}{p} - \frac{1}{2}}. \end{aligned} \tag{20}$$

Combining (18), (19) and (20), we obtain the upper estimate for the quantity $e_m(\mathbf{H}^{\omega}_p)_{B_{q,1}}$, and hence for $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$, $1 \le \theta < \infty$. The lower estimate in (16) is a consequence of Theorem B and the relation $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_q$. $\square$

Remark 2. *If $\omega(t) = t^r$, $r > \frac{d}{p}$, $1 < p \le 2 < q < \infty$, $1 \le \theta \le \infty$, then from* (16) *we obtain the following relation*

$$e_m(\mathbf{B}^r_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d} + \frac{1}{p} - \frac{1}{2}}.$$

The above relation was established in the work [42].

At the end of this part of the work, we formulate and prove a statement that is a consequence of Theorem 2 and Theorem B.

Theorem 3. *Let $2 \le p < q < \infty$, $1 \le \theta \le \infty$, and $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > \frac{d}{2}$ and the condition (S_l). Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp \omega\big(m^{-\frac{1}{d}}\big) \tag{21}$$

is valid.

Proof. The upper estimate in (21) follows from Theorem 2 for $p = 2$, due to the embedding $\mathbf{B}^{\omega}_{p,\theta} \subset \mathbf{B}^{\omega}_{2,\theta}$, $2 \le p < \infty$. Therefore, according to (16) we have

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \ll e_m(\mathbf{B}^{\omega}_{2,\theta})_{B_{q,1}} \asymp \omega\big(m^{-\frac{1}{d}}\big).$$

The corresponding lower estimate in (21) is a consequence of Theorem B and the relation $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_q$, i.e.

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \gg e_m(\mathbf{B}^{\omega}_{p,\theta})_q \asymp \omega\big(m^{-\frac{1}{d}}\big).$$

□

Remark 3. *If $\omega(t) = t^r$, $r > \frac{d}{2}$, $2 \le p < q < \infty$, $1 \le \theta \le \infty$, then the following relation*

$$e_m(\mathbf{B}^{r}_{p,\theta})_{B_{q,1}} \asymp m^{-\frac{r}{d}}.$$

holds.

The above estimate was established in the work [23].

3 Comments

First, let us recall the definition of the approximation characteristic, which is close to the best m-term trigonometric approximation $e_m(F)_X$.

For $f \in X$, we denote

$$S_{\Theta_m}(f) := S_{\Theta_m}(f,x) = \sum_{k\in\Theta_m} \widehat{f}(k)e^{i(k,x)}$$

and consider the quantity

$$e^{\perp}_m(f)_X := \inf_{\Theta_m} \|f - S_{\Theta_m}(f)\|_X.$$

If $F \subset X$ is a functional class, then we put

$$e^{\perp}_m(F)_X := \sup_{f\in F} e^{\perp}_m(f)_X.$$

The quantity $e^{\perp}_m(F)_X$ is called the best orthogonal trigonometric approximation of the class F in the space X. The quantities $e^{\perp}_m(F)_X$ for different functional classes F in the Lebesgue spaces $L_q(\mathbb{T}^d)$ as well as in some of their subspaces were investigated in many papers (see, e.g., [9, 15, 16, 25, 29, 35, 37, 38]), where the interested reader can find a more detailed bibliography.

Note, that from the definitions of the quantities $e_m(F)_X$ and $e^{\perp}_m(F)_X$ we get the following relation

$$e_m(F)_X \le e^{\perp}_m(F)_X.$$

Let us formulate two well-known statements with which it is worth comparing the results of Theorems 1–3.

Theorem C ([15]). *Let $1 \leq p, q, \theta \leq \infty$, $(p, q) \notin \{(1,1), (\infty, \infty)\}$, and $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > d\big(\frac{1}{p} - \frac{1}{q}\big)_+$ and the condition (S_l). Then for any $m \in \mathbb{N}$ the following estimate*

$$e^{\perp}_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp \omega\big(m^{-\frac{1}{d}}\big) m^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} \tag{22}$$

is valid.

Corollary A ([15]). *Let $1 \leq \theta \leq \infty$, $1 \leq p \leq q \leq 2$ or $1 \leq q \leq p \leq \infty$, and $\omega(t)$ satisfies the condition (S^{α}) with some $\alpha > d\big(\frac{1}{p} - \frac{1}{q}\big)_+$ and the condition (S_l). Then for any $m \in \mathbb{N}$ the following estimate*

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp \omega\big(m^{-\frac{1}{d}}\big) m^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}$$

holds.

So, moving directly to the comments, we note the following.

a) Comparing the results of Theorems 1–3 with the estimate (22), for the respective values of the parameters p, q, θ and α, we observe differences in the orders of quantities $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$ and $e^{\perp}_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$.

   In addition, by studying the quantities $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$, we discovered the so-called "small smoothness" effect, which consists in the following. When the parameter α crosses the limiting case $\alpha = \frac{d}{p}$, a "jump" in the estimate of the quantity $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$ appears. In other words, the estimates of this quantity for $d\big(\frac{1}{p} - \frac{1}{q}\big) < \alpha < \frac{d}{p}$ (Theorem 1) and $\alpha > \frac{d}{p}$ (Theorem 2) differ in order.

   Note that when studying the quantities $e^{\perp}_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$ (Theorem C), no such effect was observed.

b) The orders of quantities considered in Theorem C and Corollary A are realized by approximating the classes $\mathbf{B}^{\omega}_{p,\theta}$ by the trigonometric polynomials with the spectrum in cubic regions. In connection with this circumstance, we note that the orders of quantities $e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}}$, which are obtained in Theorems 1–3 by the mentioned polynomials, are not realized.

c) Analyzing the results of Theorems 1–3, Theorem B and Corollary 1, we conclude that for the corresponding values of the parameters p, q, θ and α the following relation

$$e_m(\mathbf{B}^{\omega}_{p,\theta})_{B_{q,1}} \asymp e_m(\mathbf{B}^{\omega}_{p,\theta})_q$$

   holds.

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Федуник-Яремчук О.В., Гембарська С.Б., Романюк І.А. *Найкращі m-членні тригонометричні наближення ізотропних класів типу Нікольського-Бєсова періодичних функцій багатьох змінних* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 67–81.

Одержано точні за порядком оцінки найкращих m-членних наближень ізотропних класів типу Нікольського-Бєсова $B^{\omega}_{p,\theta}$ періодичних функцій багатьох змінних у просторах $B_{q,1}$ при $1 < p < q < \infty$, $q \geq 2$. Особливістю цих просторів, як лінійних підпросторів L_q, є те, що норма в них є сильнішою, ніж L_q-норма. Виявлено, що одержані оцінки розглянутої апроксимаційної характеристики співпадають за порядком з оцінками відповідної характеристики класів $B^{\omega}_{p,\theta}$ у просторах L_q.

Ключові слова і фрази: періодична функція багатьох змінних, клас типу Нікольського-Бєсова, найкраще m-членне тригонометричне наближення.