ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (1), 42–66
doi:10.15330/cmp.17.1.42-66

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.42–66

Estimates for sums of Dirichlet series

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In the article, we prove approximation theorems that allow us to estimate, with sufficient accuracy, the supremum modulus of a Dirichlet series by the maximal term of another Dirichlet series associated with the given one. Using these theorems, we establish necessary and sufficient conditions on the sequence of coefficients of a Dirichlet series, under which the most general asymptotic and global estimates from above for its supremum modulus hold.

Key words and phrases: analytic function, entire function, Dirichlet series, abscissa of absolute convergence, supremum modulus, maximal term, Young conjugate function.

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Introduction

For an analytic function presented by a Dirichlet series with a nonnegative, increasing to $+\infty$ sequence of exponents, the following problem is classical: find conditions on the sequence of coefficients and the sequence of exponents of the series under which for the supremum modulus of the function on a vertical line one or another estimate from above holds. This problem is of a general nature and various approaches and methods have been proposed for its solution in works of many authors.

In the case when the sequence of exponents of a Dirichlet series coincides with the sequence of nonnegative integers, the considered problem is equivalent to the following problem for a power series: find conditions on the sequence of its coefficients under which for the maximum modulus of its sum on a circle one or another estimate from above holds. Classical methods, that allow obtaining such conditions for a power series and are based on the technique of its maximal term, are the Wiman-Valiron method and the Rosenbloom probabilistic method (see [26, 31, 34] and [45, Chapter IX]). In works of M.M. Sheremeta (see [37] and the bibliography there), the Wiman-Valiron method was modified to study properties of entire (absolutely convergent in $\mathbb{C}$) Dirichlet series. For the same purpose, an adaptation of the Rosenbloom probabilistic method was carried out in the work of O.B. Skaskiv [41]. As a result, for an entire Dirichlet series, it was possible to find necessary and sufficient conditions on the sequence of its exponents, under which the most general estimates from above for the supremum modulus of its sum by its maximal term are satisfied.

A typical feature of the estimates obtained by the Wiman-Valiron and Rosenbloom methods or their modifications is that, under the conditions found for the exponents, these estimates are

УДК 517.53
2020 *Mathematics Subject Classification:* 30B50, 30B10, 30B30, 30C80, 30D15, 30D20.

satisfied only outside certain exceptional sets. Questions about the existence and the sizes of exceptional sets in some estimates between the modulus of the sum of a series and its maximal term were considered, for instance, in the articles [13, 15, 16, 42, 43]. Conditions, under which the most general estimates from above for the supremum modulus of an entire Dirichlet series by its maximal term hold without exceptional sets, are found in the works [14, 36], and similar problems for a Dirichlet series absolutely convergent in a half-plane was investigated in the work [44].

A slightly different approach for estimating the sums of entire Dirichlet series was carried out in the work of M.M. Sheremeta [38]. Actually, M.M. Sheremeta [38] established estimates for the supremum modulus of an entire Dirichlet series by the maximal term of another Dirichlet series associated with the given series (see Theorem D below). Note that the obtained estimates hold without any assumptions about the system of exponents of the given series. Another important point is that the maximum term of the associated series in a certain sense well approximates the supremum modulus of the given series. Using these facts, in [38] necessary and sufficient conditions, under which some global estimates for the sums of entire Dirichlet series hold, were established. The results from [38] were also applied in the works [22,28,33] to study other properties of entire Dirichlet series. Analogs of the results from [38] for Dirichlet series absolutely convergent in a half-plane were obtained in [23]. This article is devoted to the development of the approach proposed in [38] and its applications.

1 Definitions and previous results

Denote by $\mathbb{N}_0$ the set of all nonnegative integers, and by Λ denote the class of all nonnegative sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ increasing to $+\infty$.

Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ. Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{1}$$

and denote by $\sigma_a(F)$ the abscissa of absolute convergence of series (1). Put

$$\sigma_e(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

It is easy to see that if $\sigma_e(F) > -\infty$ and $\sigma < \sigma_e(F)$, then $|a_n|e^{\sigma\lambda_n} \to 0$ as $n \to \infty$. Therefore, for such σ, the maximum term

$$\mu(\sigma, F) = \max\left\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\right\}$$

of series (1) is defined. If $\sigma_a(F) > -\infty$, then for each $\sigma < \sigma_a(F)$ we set

$$M(\sigma, F) = \sup\left\{|F(\sigma + it)| : t \in \mathbb{R}\right\}, \quad \mathfrak{M}(\sigma, F) = \sum_{n=0}^{\infty} |a_n|e^{\sigma\lambda_n}.$$

Note that for all such σ we have $\mu(\sigma, F) \le M(\sigma, F) \le \mathfrak{M}(\sigma, F)$, and $M(\sigma, F) = \mathfrak{M}(\sigma, F) = F(\sigma)$ in the case when $a_n \ge 0$ for any $n \in \mathbb{N}_0$.

Suppose that series (1) is absolutely convergent at the point $s=0$. Put

$$R_n=\sum_{k=n}^{\infty}|a_k|,\quad n\in\mathbb{N}_0,\tag{2}$$

and along with series (1) consider the Dirichlet series

$$F_1(s)=\sum_{n=0}^{\infty}R_ne^{s\lambda_n}.\tag{3}$$

Then, as is well known (see, for example, [32, Theorem I.2.8]), $\sigma_a(F)=\sigma_e(F_1)$. Set

$$S_n=\sum_{k=0}^{n}|a_k|,\quad n\in\mathbb{N}_0,\tag{4}$$

and consider the Dirichlet series

$$F_2(s)=\sum_{n=0}^{\infty}S_ne^{s\lambda_n}.\tag{5}$$

If series (1) absolutely diverges at the point $s=0$, then, as is well known (see, for example, [32, Theorem I.2.8]), $\sigma_a(F)=\min\{0,\sigma_e(F_2)\}$.

For every fixed $A\in(-\infty,+\infty]$, by $\mathcal{D}_A(\lambda)$ we denote the class of all Dirichlet series of the form (1) such that $\sigma_a(F)\ge A$ and $a_n\lambda_n\ne 0$ for at least one value $n\in\mathbb{N}_0$, and denote by $\mathcal{D}_A^*(\lambda)$ the class of all Dirichlet series of the form (1) for which $\beta(F)\ge A$ and $a_n\lambda_n\ne 0$ for at least one value $n\in\mathbb{N}_0$. We put $\mathcal{D}_A=\cup_{\lambda\in\Lambda}\mathcal{D}_A(\lambda)$ and $\mathcal{D}_A^*=\cup_{\lambda\in\Lambda}\mathcal{D}_A^*(\lambda)$.

By X we denote the class of all functions $\alpha:\mathbb{R}\to\overline{\mathbb{R}}$. For a function $\alpha\in X$ let $\widetilde{\alpha}$ be the Young conjugate function of α, i.e.

$$\widetilde{\alpha}(x)=\sup\{x\sigma-\alpha(\sigma):\sigma\in\mathbb{R}\},\quad x\in\mathbb{R}.$$

For each function $\alpha\in X$, we set $D_\alpha=\{\sigma\in\mathbb{R}:\alpha(\sigma)<+\infty\}$. If $A\in(-\infty,+\infty]$ is fixed, then by X_A we denote the class of all functions $\alpha\in X$ for which $D_\alpha\subset(-\infty,A)$. Note that $X_{+\infty}=X$. Let Ω_A be the class of all functions $\Phi\in X_A$ such that D_Φ is an interval of the form $[a,A)$, $a<A$, Φ is continuous on D_Φ, and the following condition holds: $x\sigma-\Phi(\sigma)\to-\infty$ as $\sigma\uparrow A$ for every fixed $x\in\mathbb{R}$. In the case $A<+\infty$, the indicated condition is equivalent to the condition $\Phi(\sigma)\to+\infty$ as $\sigma\to A-0$, and in the case $A=+\infty$, it is equivalent to the condition $\Phi(\sigma)/\sigma\to+\infty$ as $\sigma\to+\infty$.

Necessary and sufficient conditions, under which some asymptotic estimates from above for the supremum modulus of a Dirichlet series hold, were found in [16, 18, 29, 30].

Theorem A ([29]). *Let $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ, $A\in(-\infty,+\infty]$, $\Phi\in\Omega_A$, and $T_0\ge t_0\ge 0$ be arbitrary constants. For each Dirichlet series $F\in\mathcal{D}_A(\lambda)$ such that $\ln\mu(\sigma,F)\le(t_0+o(1))\Phi(\sigma)$ as $\sigma\uparrow A$ we have $\ln M(\sigma,F)\le(T_0+o(1))\Phi(\sigma)$ as $\sigma\uparrow A$ if and only if*

$$\forall\, T>T_0\ \exists\, c\in(t_0,T):\ \varlimsup_{n\to\infty}\frac{\ln n}{c\widetilde{\Phi}(\lambda_n/c)-T\widetilde{\Phi}(\lambda_n/T)}<1.\tag{6}$$

Theorem B ([30]). *Let $\lambda=(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ, $\Phi\in\Omega_{+\infty}$, and $T_0\ge t_0\ge 0$ be arbitrary constants. For each Dirichlet series $F\in\mathcal{D}_{+\infty}(\lambda)$ such that $\ln\mu(\sigma,F)\le\Phi((t_0+o(1))\sigma)$ as $\sigma\to+\infty$ we have $\ln M(\sigma,F)\le\Phi((T_0+o(1))\sigma)$ as $\sigma\to+\infty$ if and only if*

$$\forall\, T>T_0\ \exists\, c\in(t_0,T):\ \varlimsup_{n\to\infty}\frac{\ln n}{\widetilde{\Phi}(\lambda_n/c)-\widetilde{\Phi}(\lambda_n/T)}<1.\tag{7}$$

The sufficiency of conditions (6) and (7) in Theorems A and B can be easily justified by using the following theorem, which gives sufficient conditions in order that the most general asymptotic estimates from above for the supremum modulus of a Dirichlet series hold.

Theorem C ([29]). *Let* $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ *be a sequence from the class* Λ, $A \in (-\infty, +\infty]$, *and* $\Phi, \Gamma \in \Omega_A$. *If*

$$\sum_{n=0}^{\infty} \frac{1}{e^{\widetilde{\Phi}(\lambda_n)-\widetilde{\Gamma}(\lambda_n)}} < +\infty,$$

then every Dirichlet series F *from the class* $\mathcal{D}_A^*(\lambda)$ *such that* $\ln \mu(\sigma, F) \le \Phi(\sigma)$, $\sigma \in [\sigma_1, A)$, *belongs to the class* $\mathcal{D}_A$ *and for it we have* $\ln M(\sigma, F) \le \Gamma(\sigma)$, $\sigma \in [\sigma_2, A)$.

An analysis of the proofs of Theorems A and B given in [29] and [30], respectively, shows that the main and nontrivial parts in these proofs are the justifications of the necessity of conditions (6) and (7). Actually, these justifications are of a constructive nature. It is clear that Theorem C cannot be used for this. The following two theorems are much more effective in this regard, and their application does not require any considerations of a constructive nature.

Theorem D ([38]). *Let* $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ *be a sequence from the class* Λ, *and* F *be a Dirichlet series of the form* (1). *Then:*

(i) if the series F *is absolutely convergent at the point* $s = 0$, *then* $F \in \mathcal{D}_{+\infty}$ *if and only if* $F_1 \in \mathcal{D}^*_{+\infty}$, *where* F_1 *is the series defined by* (3) *and* (2);

(ii) if $F \in \mathcal{D}_{+\infty}$, *then for arbitrary* $\sigma \ge 0$ *and* $\varepsilon > 0$ *we have*

$$\mu(\sigma, F_1) \le \mathfrak{M}(\sigma, F) \le \mu(\sigma+\varepsilon, F_1)\frac{\sigma+\varepsilon}{\varepsilon}. \tag{8}$$

Theorem E ([23]). *Let* $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ *be a sequence from the class* Λ, *and* F *be a Dirichlet series of the form* (1). *Then:*

(i) $F \in \mathcal{D}_0$ *if and only if* $F_2 \in \mathcal{D}_0^*$, *where* F_2 *is the series defined by* (5) *and* (4);

(ii) if $F \in \mathcal{D}_0$, *then for arbitrary* $\sigma < 0$ *and* $\delta \in (0,1)$ *we have*

$$\mu(\sigma, F_2) \le \mathfrak{M}(\sigma, F) \le \frac{\mu(\delta\sigma, F_2)}{(1-\delta)e^{(1-\delta)|\sigma|\lambda_0}}. \tag{9}$$

Theorems D and E can be used to establish necessary and sufficient conditions under which practically all estimates for sums of Dirichlet series considered in this paper hold. Then such conditions will be conditions on the sequence of remainders $(R_n)_{n\in\mathbb{N}_0}$ and partial sums $(S_n)_{n\in\mathbb{N}_0}$ of the series $\sum_{k=0}^{\infty} |a_k|$, respectively, and therefore these conditions, especially in the case of entire Dirichlet series, can be difficult to verify. In the next section, we establish more flexible theorems that will allow us to obtain conditions under which the most general estimates for sums of Dirichlet series hold in a simpler form.

2 Main results

Let $c > 0$ be a constant, F be an arbitrary Dirichlet series of the form (1), and let

$$N = \min\{n \in \mathbb{N}_0 : a_n \neq 0\}. \tag{10}$$

For each $n \in \mathbb{N}_0$ we set

$$b_n = 0, \quad \text{if} \quad n < N; \quad b_n = \sum_{\lambda_n \le \lambda_k < \lambda_n + c} |a_k|, \quad \text{if} \quad n \ge N, \tag{11}$$

and consider the Dirichlet series

$$G_c(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n}. \tag{12}$$

Theorem 1. *Let $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ, F be an arbitrary Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12), (11), *and* (10). *Then:*

(i) $F \in \mathcal{D}_{+\infty}$ if and only if $G_c \in \mathcal{D}^*_{+\infty}$;

(ii) if $F \in \mathcal{D}_{+\infty}$, then for arbitrary $\sigma \ge 0$ and $\varepsilon > 0$ we have

$$\mu(\sigma, G_c) \le \mathfrak{M}(\sigma, F) \le \frac{\mu(\sigma+\varepsilon, G_c)}{e^{\varepsilon\lambda_N}}\Big(\frac{e^{\varepsilon c}}{e^{\varepsilon c}-1} + \frac{\sigma}{\varepsilon}\Big). \tag{13}$$

Proof. Let $F \in \mathcal{D}_{+\infty}$. Then for the series F_1 defined by (3) and (2), we get $\sigma_e(F_1) = \sigma_a(F) = +\infty$. Since $\sigma_e(G_c) \ge \sigma_e(F_1)$, we have $\sigma_e(G_c) = +\infty$, and, therefore, $G_c \in \mathcal{D}^*_{+\infty}$.

Suppose that $G_c \in \mathcal{D}^*_{+\infty}$. By this assumption, the maximal term $\mu(\sigma, G_c)$ is defined for all $\sigma \in \mathbb{R}$. Let $\sigma \ge 0$ and $\varepsilon > 0$. We prove that then the right inequality in (13) holds (this, in particular, will imply that $F \in \mathcal{D}_{+\infty}$).

Let $l \ge \lambda_N$ be a fixed number such that the interval $[l, l+c)$ contains at least one term of the sequence λ. We put

$$r = \min\{k \in \mathbb{N}_0 : \lambda_k \in [l, l+c)\}, \quad p = \max\{k \in \mathbb{N}_0 : \lambda_k \in [l, l+c)\}, \tag{14}$$

and prove the inequality

$$\sum_{l \le \lambda_k < l+c} |a_k| e^{\sigma\lambda_k} \le \mu(\sigma+\varepsilon, G_c)\Big(\frac{1}{e^{\varepsilon\lambda_r}} + \sigma\int_{\lambda_r}^{\lambda_p} \frac{dt}{e^{\varepsilon t}}\Big). \tag{15}$$

This inequality is obvious if $p = r$. Consider the case when $p > r$. In this case, for each integer $m \in [r, p]$ we set $s_m = \sum_{k=m}^{p} |a_k|$. Noting that $s_m \le b_m$ for all integers $m \in [r, p]$, we obtain

$$\begin{aligned}
\sum_{l \le \lambda_k < l+c} |a_k| e^{\sigma\lambda_k} &= \sum_{m=r}^{p} |a_m| e^{\sigma\lambda_m} = \sum_{m=r}^{p-1} (s_m - s_{m+1}) e^{\sigma\lambda_m} + s_p e^{\sigma\lambda_p} \\
&= s_r e^{\sigma\lambda_r} + \sum_{m=r+1}^{p} s_m (e^{\sigma\lambda_m} - e^{\sigma\lambda_{m-1}}) \le b_r e^{\sigma\lambda_r} + \sum_{m=r+1}^{p} b_m (e^{\sigma\lambda_m} - e^{\sigma\lambda_{m-1}}) \\
&= b_r e^{\sigma\lambda_r} + \sigma \sum_{m=r+1}^{p} b_m \int_{\lambda_{m-1}}^{\lambda_m} e^{\sigma t} dt \le \frac{b_r e^{(\sigma+\varepsilon)\lambda_r}}{e^{\varepsilon\lambda_r}} + \sigma \sum_{m=r+1}^{p} b_m e^{(\sigma+\varepsilon)\lambda_m} \int_{\lambda_{m-1}}^{\lambda_m} \frac{dt}{e^{\varepsilon t}} \\
&\le \mu(\sigma+\varepsilon, G_c)\Big(\frac{1}{e^{\varepsilon\lambda_r}} + \sigma \sum_{m=r+1}^{p} \int_{\lambda_{m-1}}^{\lambda_m} \frac{dt}{e^{\varepsilon t}}\Big) = \mu(\sigma+\varepsilon, G_c)\Big(\frac{1}{e^{\varepsilon\lambda_r}} + \sigma\int_{\lambda_r}^{\lambda_p} \frac{dt}{e^{\varepsilon t}}\Big),
\end{aligned}$$

that is, (15) holds.

Let us introduce for each $m \in \mathbb{N}_0$ the notation $l_m = \lambda_N + mc$. Noting that $l_{m+1} = l_m + c$ for all $m \in \mathbb{N}_0$, and using (15) and (14), we obtain

$$\begin{aligned}\mathfrak{M}(\sigma, F) &= \sum_{m=0}^{\infty} \sum_{l_m \le \lambda_k < l_{m+1}} |a_k| e^{\sigma\lambda_k} = \sum_{m=0}^{\infty} \sum_{l_m \le \lambda_k < l_m + c} |a_k| e^{\sigma\lambda_k} \\ &\le \sum_{m=0}^{\infty} \mu(\sigma+\varepsilon, G_c)\Big(\frac{1}{e^{\varepsilon l_m}} + \sigma \int_{l_m}^{l_{m+1}} \frac{dt}{e^{\varepsilon t}}\Big) = \mu(\sigma+\varepsilon, G_c)\Big(\sum_{m=0}^{\infty} \frac{1}{e^{\varepsilon(\lambda_N + mc)}} + \sigma \int_{\lambda_N}^{+\infty} \frac{dt}{e^{\varepsilon t}}\Big) \\ &= \mu(\sigma+\varepsilon, G_c)\Big(\frac{e^{\varepsilon c}}{e^{\varepsilon\lambda_N}(e^{\varepsilon c}-1)} + \frac{\sigma}{\varepsilon e^{\varepsilon\lambda_N}}\Big) = \frac{\mu(\sigma+\varepsilon, G_c)}{e^{\varepsilon\lambda_N}}\Big(\frac{e^{\varepsilon c}}{e^{\varepsilon c}-1} + \frac{\sigma}{\varepsilon}\Big).\end{aligned}$$

Since the inequality $\mathfrak{M}(\sigma, F) \ge \mu(\sigma, G_c)$ is obvious, the theorem is completely proved. □

Remark 1. *The right inequality in* (8) *can be obtained from Theorem 1. In fact, since* $\mu(\sigma, G_c) \le \mu(\sigma, F_1)$ *for any* $c > 0$, *by* (13) *for each fixed* $\sigma \ge 0$ *we have*

$$\mathfrak{M}(\sigma, F) \le \mu(\sigma+\varepsilon, F_1)\Big(\frac{e^{\varepsilon c}}{e^{\varepsilon c}-1} + \frac{\sigma}{\varepsilon}\Big).$$

It remains to direct c *here to* $+\infty$.

Next, we will prove an analogue of Theorem 1 for Dirichlet series absolutely convergent in a half-plane.

Let $c > 0$ be a constant, F be an arbitrary Dirichlet series of the form (1), and N be the number defined by (10). For each $n \in \mathbb{N}_0$, we set

$$b_n = 0, \quad \text{if} \quad n < N; \quad b_n = \sum_{\lambda_n - c < \lambda_k \le \lambda_n} |a_k|, \quad \text{if} \quad n \ge N. \tag{16}$$

Theorem 2. *Let* $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ *be a sequence from the class* Λ, F *be an arbitrary Dirichlet series of the form* (1), $c > 0$, *and* G_c *be the series defined by* (12), (16), *and* (10). *Then:*

(i) $F \in \mathcal{D}_0$ *if and only if* $G_c \in \mathcal{D}_0^*$;

(ii) if $F \in \mathcal{D}_0$, *then for arbitrary* $\sigma < 0$ *and* $\varepsilon \in (0, -\sigma)$ *we have*

$$\mu(\sigma, G_c) \le \mathfrak{M}(\sigma, F) \le \frac{\mu(\sigma+\varepsilon, G_c)}{e^{\varepsilon\lambda_N}}\Big(\frac{|\sigma|}{\varepsilon} + \frac{1}{e^{\varepsilon(\lambda_L - \lambda_N)}} + \frac{1}{e^{\varepsilon c}}\Big), \tag{17}$$

where $L = L(c) := \max\{n \in \mathbb{N}_0 : \lambda_n < \lambda_N + c\}$.

Proof. Let $F \in \mathcal{D}_0$, i.e. $\sigma_a(F) \ge 0$. Then by Theorem E, for the series F_2 defined by (5) and (4), we obtain $\sigma_e(F_2) \ge 0$. Since $\sigma_e(G_c) \ge \sigma_e(F_2)$, we have $\sigma_e(G_c) \ge 0$, and therefore $G_c \in \mathcal{D}_0^*$. Note that the inequality $\sigma_e(F_2) \ge 0$ is easy to prove directly by assuming the contrary: if $\sigma_e(F_2) < 0$, then the series F absolutely diverges at the point $s = 0$, and then, as noted above, the equality $\sigma_a(F) = \min\{0, \sigma_e(F_2)\}$ should hold, which is impossible.

Now suppose that $G_c \in \mathcal{D}_{+\infty}^*$. By this assumption, the maximal term $\mu(\sigma, G_c)$ is defined for all $\sigma < 0$. Let $\sigma < 0$ and $\varepsilon \in (0, -\sigma)$. We prove that then the right inequality in (17) holds (this, in particular, implies that $F \in \mathcal{D}_0$).

We fix an arbitrary number $l \ge \lambda_N$ such that the interval $[l, l+c)$ contains at least one term of the sequence λ. Define r and p by (14) and prove the inequality

$$\sum_{l \le \lambda_k < l+c} |a_k| e^{\sigma\lambda_k} \le \mu(\sigma+\varepsilon, G_c)\Big(|\sigma| \int_{\lambda_r}^{\lambda_p} \frac{dt}{e^{\varepsilon t}} + \frac{1}{e^{\varepsilon\lambda_p}}\Big). \tag{18}$$

If $p = r$, then this inequality is obvious. Consider the case when $p > r$. In this case, for each integer $m \in [r, p]$, we set $s_m = \sum_{k=r}^{m} |a_k|$ and note that $s_m \le b_m$. Therefore

$$\begin{aligned}\sum_{l\le\lambda_k<l+c} |a_k|e^{\sigma\lambda_k} &= \sum_{m=r}^{p} |a_m|e^{\sigma\lambda_m} = s_r e^{\sigma\lambda_r} + \sum_{m=r+1}^{p} (s_m - s_{m-1})e^{\sigma\lambda_m}\\ &= \sum_{m=r}^{p-1} s_m(e^{\sigma\lambda_m} - e^{\sigma\lambda_{m+1}}) + s_p e^{\sigma\lambda_p} \le \sum_{m=r}^{p-1} b_m(e^{\sigma\lambda_m} - e^{\sigma\lambda_{m+1}}) + b_p e^{\sigma\lambda_p}\\ &= |\sigma|\sum_{m=r}^{p-1} b_m \int_{\lambda_m}^{\lambda_{m+1}} e^{\sigma t}dt + b_p e^{\sigma\lambda_p} \le |\sigma|\sum_{m=r}^{p-1} b_m e^{(\sigma+\varepsilon)\lambda_m} \int_{\lambda_m}^{\lambda_{m+1}} \frac{dt}{e^{\varepsilon t}}dt + \frac{b_p e^{(\sigma+\varepsilon)\lambda_p}}{e^{\varepsilon\lambda_p}}\\ &\le \mu(\sigma+\varepsilon, G_c)\Big(|\sigma|\sum_{m=r}^{p-1}\int_{\lambda_m}^{\lambda_{m+1}} \frac{dt}{e^{\varepsilon t}}dt + \frac{1}{e^{\varepsilon\lambda_p}}\Big) = \mu(\sigma+\varepsilon, G_c)\Big(|\sigma|\int_{\lambda_r}^{\lambda_p} \frac{dt}{e^{\varepsilon t}} + \frac{1}{e^{\varepsilon\lambda_p}}\Big),\end{aligned}$$

that is, (18) holds.

Setting $l_m = \lambda_N + mc$ for each $m \in \mathbb{N}_0$ and using (18) and (14), we have

$$\begin{aligned}\mathfrak{M}(\sigma, F) &= \sum_{m=0}^{\infty} \sum_{l_m\le\lambda_k<l_{m+1}} |a_k|e^{\sigma\lambda_k} = \sum_{\lambda_N\le\lambda_k<\lambda_N+c} |a_k|e^{\sigma\lambda_k} + \sum_{m=1}^{\infty} \sum_{l_m\le\lambda_k<l_m+c} |a_k|e^{\sigma\lambda_k}\\ &\le \mu(\sigma+\varepsilon, G_c)\Big(|\sigma|\int_{\lambda_N}^{l_1} \frac{dt}{e^{\varepsilon t}} + \frac{1}{e^{\varepsilon\lambda_L}}\Big) + \sum_{m=1}^{\infty} \mu(\sigma+\varepsilon, G_c)\left(|\sigma|\int_{l_m}^{l_{m+1}} \frac{dt}{e^{\varepsilon t}} + \frac{1}{e^{\varepsilon l_m}}\right)\\ &= \mu(\sigma+\varepsilon, G_c)\Big(|\sigma|\int_{\lambda_N}^{+\infty} \frac{dt}{e^{\varepsilon t}} + \frac{1}{e^{\varepsilon\lambda_L}} + \sum_{m=1}^{\infty} \frac{1}{e^{\varepsilon(\lambda_N+mc)}}\Big)\\ &= \mu(\sigma+\varepsilon, G_c)\Big(\frac{|\sigma|}{\varepsilon e^{\varepsilon\lambda_N}} + \frac{1}{e^{\varepsilon\lambda_L}} + \frac{1}{e^{\varepsilon\lambda_N}(e^{\varepsilon c}-1)}\Big),\end{aligned}$$

that is, the right inequality in (17) holds. Since the left inequality in (17) is trivial, the theorem is completely proved. □

Remark 2. *Let $\sigma < 0$ and $\delta \in (0, 1)$ be arbitrary numbers. Put $\varepsilon = (1-\delta)|\sigma|$. Then the right inequality in* (17) *can be rewritten as*

$$\mathfrak{M}(\sigma, F) \le \frac{\mu(\delta\sigma, G_c)}{e^{(1-\delta)|\sigma|\lambda_N}}\Big(\frac{1}{1-\delta} + \frac{1}{e^{(1-\delta)|\sigma|(\lambda_{L(c)}-\lambda_N)}} + \frac{1}{e^{(1-\delta)|\sigma|c}-1}\Big). \tag{19}$$

This implies the right inequality in (9)*. In fact, since $\mu(\delta\sigma, G_c) \le \mu(\delta\sigma, F_2)$ for any $c > 0$, it suffices in* (19) *to first replace $\mu(\delta\sigma, G_c)$ with $\mu(\delta\sigma, F_2)$, and then direct c to $+\infty$.*

Remark 3. *Theorems 1 and 2 can be used to estimate remainders of Dirichlet series. Estimates of this kind are needed, for example, when establishing Berstein-type inequalities for the Dirichlet series and its derivative (see [6, 10–12, 35, 40]). In particular, if $F \in \mathcal{D}_{+\infty}$, then by Theorem 1 for arbitrary $K \in \mathbb{N}_0$, $\sigma \ge 0$, and $\varepsilon > 0$ we have*

$$\sum_{n\ge K} |a_n|e^{\sigma\lambda_n} \le \frac{\mu(\sigma+\varepsilon, G_c)}{e^{\varepsilon\lambda_K}}\Big(\frac{e^{\varepsilon c}}{e^{\varepsilon c}-1} + \frac{\sigma}{\varepsilon}\Big).$$

3 Auxiliary results

In this section, we give some simple and well-known auxiliary statements describing properties of Young conjugate functions, as well as the growth of the logarithm of the maximal term of a Dirichlet series in terms of such functions. Using these statements, in the following sections we give numerous applications of Theorems 1, E and 2.

First of all, we note that if $\alpha \in X$, then $\widetilde{\alpha}$ is a convex function, i.e. for arbitrary $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 < x_2 < x_3$, we have

$$\widetilde{\alpha}(x_2)(x_3 - x_1) \le \widetilde{\alpha}(x_1)(x_3 - x_2) + \widetilde{\alpha}(x_3)(x_2 - x_1).$$

Moreover, $\widetilde{\widetilde{\alpha}}(\sigma) \le \alpha(\sigma)$ for all $\sigma \in \mathbb{R}$ (see, for example, [29]).

It is also easy to see that for arbitrary functions $\alpha, \gamma \in X$ such that $\alpha(\sigma) \le \gamma(\sigma)$ for all $\sigma \in \mathbb{R}$, we have $\widetilde{\alpha}(x) \ge \widetilde{\gamma}(x)$ for all $x \in \mathbb{R}$, and therefore $\widetilde{\widetilde{\alpha}}(\sigma) \le \widetilde{\widetilde{\gamma}}(\sigma)$ for all $\sigma \in \mathbb{R}$.

Lemma 1 ([27]). *Let $\alpha \in X$ be a function such that D_α is an interval of the real axis and α is convex on D_α. Then $\widetilde{\widetilde{\alpha}}(\sigma) = \alpha(\sigma)$ for all $\sigma \in D_\alpha$.*

Lemma 2 ([27]). *Let $\alpha, \gamma \in X$ be functions such that D_γ is an interval of the real axis, $\alpha(\sigma) \le \gamma(\sigma)$ for all $\sigma \in D_\gamma$, and α is convex on D_γ. Then $\alpha(\sigma) \le \widetilde{\widetilde{\gamma}}(\sigma)$ for all $\sigma \in D_\gamma$.*

Since $\widetilde{\widetilde{\gamma}}(\sigma) \le \gamma(\sigma)$ for all $\sigma \in D_\gamma$, Lemma 2 defines the geometric meaning of the second Young conjugate function: among all functions α convex on D_γ and such that $\alpha(\sigma) \le \gamma(\sigma)$ for all $\sigma \in D_\gamma$, the function $\widetilde{\widetilde{\gamma}}$ takes on the largest possible value at each point of D_γ. Actually, using Lemmas 1 and 2 and noting that the maximum of two convex functions on some interval is also a convex function on this interval, it is easy to substantiate the following statement by geometric considerations.

Lemma 3. *Let $A \in (-\infty, +\infty]$, $\alpha \in X_A$, $D_\alpha = (-\infty, A)$, $\beta \in \Omega_A$, $D_\beta = [a, A)$, and the functions α and β are convex on D_α and D_β, respectively. Suppose that $\gamma(\sigma) = \alpha(\sigma)$ for all $\sigma < a$ and $\gamma(\sigma) = \max\{\alpha(\sigma), \beta(\sigma)\}$ for all $\sigma \in [a, A)$. Then:*

- *(i) if $\gamma(a) = \alpha(a)$, we have $\widetilde{\widetilde{\gamma}}(\sigma) = \gamma(\sigma)$ for all $\sigma < A$;*
- *(ii) if $\gamma(a) > \alpha(a)$, then the function $k(\sigma) = (\gamma(\sigma) - \alpha(a))/(\sigma - a)$, $\sigma \in (a, A)$, takes on a minimum value at some point $\sigma_0 \in (a, A)$, and $\widetilde{\widetilde{\gamma}}(\sigma) = \gamma(\sigma)$ for all $\sigma < A$ such that $\sigma \notin (a, \sigma_0)$, and $\widetilde{\widetilde{\gamma}}(\sigma) = k(\sigma_0)(\sigma - a) + \alpha(a)$ for all $\sigma \in [a, \sigma_0]$;*
- *(iii) $\alpha(\sigma) \le \widetilde{\widetilde{\gamma}}(\sigma)$ for all $\sigma < A$.*

Lemma 4 ([29]). *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $\varphi(x) = \max\{\sigma \in D_\Phi : x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$ for all $x \in \mathbb{R}$. Then:*

- *(i) φ is a nondecreasing function on $\mathbb{R}$;*
- *(ii) φ is continuous from the right on $\mathbb{R}$;*
- *(iii) $\varphi(x) \to A$ as $x \to +\infty$;*
- *(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at each point $x \in \mathbb{R}$;*
- *(v) if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then $\widetilde{\Phi}(x)/x$ increases to A on $(x_0, +\infty)$;*
- *(vi) the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$.*

Lemma 5 ([27]). *Let $A \in (-\infty, +\infty]$, $\Psi \in \Omega_A$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, and Γ be the second Young conjugate function of Ψ. Then $\psi(x)$ is the right-hand derivative of $\widetilde{\Gamma}(x)$, $\Gamma(\psi(x)) = \Psi(\psi(x))$, and $\Gamma'_-(\psi(x)) \le x \le \Gamma'_+(\psi(x))$ at each point $x \in \mathbb{R}$.*

Lemma 6 ([27]). *Let $A \in (-\infty, +\infty]$, and functions $\alpha, \gamma \in \Omega_A$ be such that $\alpha(\sigma) = \gamma(\sigma)$ for all $\sigma \in [\sigma_1, A)$ with some $\sigma_1 < A$. Then there exist numbers $x_0 \in \mathbb{R}$ and $\sigma_2 \in [\sigma_1, A)$ such that $\widetilde{\alpha}(x) = \widetilde{\gamma}(x)$ for all $x \ge x_0$ and $\widetilde{\widetilde{\alpha}}(\sigma) = \widetilde{\widetilde{\gamma}}(\sigma)$ for all $\sigma \in [\sigma_2, A)$.*

Lemma 7 ([27]). *Let $A \in (-\infty, +\infty]$, $\gamma \in \Omega_A$, and $F \in \mathcal{D}^*_A$ be a Dirichlet series of the form* (1). *Then the following conditions are equivalent:*

(i) there exists $\sigma_1 \in D_\gamma$ such that $\ln \mu(\sigma, F) \le \gamma(\sigma)$ for all $\sigma \in [\sigma_1, A)$;

(ii) there exists $\sigma_2 \in D_\gamma$ such that $\ln \mu(\sigma, F) \le \widetilde{\widetilde{\gamma}}(\sigma)$ for all $\sigma \in [\sigma_2, A)$;

(iii) there exists $n_0 \in \mathbb{N}_0$ such that $\ln |a_n| \le -\widetilde{\gamma}(\lambda_n)$ for all integers $n \ge n_0$.

Lemma 8 ([23]). *Let $A \in (-\infty, +\infty]$, $\gamma \in X_A$, and $F \in \mathcal{D}^*_A$ be a Dirichlet series of the form* (1). *Then the following conditions are equivalent:*

(i) $\ln \mu(\sigma, F) \le \gamma(\sigma)$ for all $\sigma < A$;

(ii) $\ln |a_n| \le -\widetilde{\gamma}(\lambda_n)$ for all $n \in \mathbb{N}_0$.

Lemma 9. *Let $A \in (-\infty, +\infty]$, $\Phi \in X_A$, a and b be positive constants, and c, d, and k be real constants. Then for the function $\beta(\sigma) = a\Phi(b\sigma + c) + d\sigma + k$, $\sigma \in \mathbb{R}$, we have $\beta \in X_{Ab+c}$ and*

$$\widetilde{\beta}(x) = a\widetilde{\Phi}\Big(\frac{x-d}{ab}\Big) - \frac{c(x-d)}{b} - k, \quad x \in \mathbb{R}.$$

Proof. The fact that $\beta \in X_{Ab+c}$ is obvious. Furthermore, using the notation $y = b\sigma + c$, for all $x \in \mathbb{R}$ we get

$$\begin{aligned} \widetilde{\beta}(x) &= \sup_{\sigma \in \mathbb{R}} (x\sigma - a\Phi(b\sigma + c) - d\sigma - k) = \sup_{y \in \mathbb{R}} \Big(\frac{x}{b}(y-c) - a\Phi(y) - \frac{d}{b}(y-c) - k\Big) \\ &= a \sup_{y \in \mathbb{R}} \Big(\frac{x-d}{ab} y - \Phi(y)\Big) - \frac{c(x-d)}{b} - k = a\widetilde{\Phi}\Big(\frac{x-d}{ab}\Big) - \frac{c(x-d)}{b} - k, \end{aligned}$$

and therefore, the lemma is proved. □

4 Asymptotic estimates for the sums of Dirichlet series

Let $F \in \mathcal{D}_A$. In this section, we establish asymptotic estimations from above for $\ln \mathfrak{M}(\sigma, F)$, provided that an asymptotic estimation from above for the logarithm of the maximal term of an associated series is known. In quite general situations, the form of an asymptotic estimation for $\ln \mathfrak{M}(\sigma, F)$ will be the same as the form of an asymptotic estimation for the logarithm of the maximal term of the associated series. In some of these situations, applying Lemma 7 to the associated series, we establish conditions on its coefficients that are necessary and sufficient to satisfy the corresponding asymptotic estimation. First, we consider the case of entire Dirichlet series.

Theorem 3. *Let $\Psi \in \Omega_{+\infty}$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If*

$$\exists\, \sigma_0 \in D_\Psi\ \forall\, \sigma \geq \sigma_0 :\ \ln \mu(\sigma, G_c) \leq \Psi(\sigma), \tag{20}$$

then for every $\varepsilon > 0$ we have

$$\ln \mathfrak{M}(\sigma, F) - \Psi(\sigma+\varepsilon) - \ln \sigma \to -\infty, \quad \sigma \to +\infty. \tag{21}$$

Proof. Let Γ be the second Young conjugate function of Ψ. Then $\Gamma(\sigma) \leq \Psi(\sigma)$ for all $\sigma \in \mathbb{R}$. Since $\Gamma \in \Omega_{+\infty}$ and Γ is convex on D_Ψ, we obtain $\Gamma'_+(\sigma) \nearrow +\infty$ as $\sigma \uparrow +\infty$, and for all $\sigma_1, \sigma_2 \in D_\Psi$ we have $\Gamma(\sigma_1) - \Gamma(\sigma_2) \geq (\sigma_1 - \sigma_2)\Gamma'_+(\sigma_2)$. In addition, by Lemma 7, the condition (20) is equivalent to the condition

$$\exists\, \sigma_1 \in D_\Psi\ \forall\, \sigma \geq \sigma_1 :\ \ln \mu(\sigma, G_c) \leq \Gamma(\sigma). \tag{22}$$

Let $\varepsilon > 0$. Fixing some $\delta \in (0, \varepsilon)$, we obtain

$$\Gamma(\sigma+\varepsilon) - \Gamma(\sigma+\delta) \geq (\varepsilon - \delta)\Gamma'_+(\sigma+\delta), \quad \sigma \in D_\Psi. \tag{23}$$

Using Theorem 1 and (22), we have

$$\ln \mathfrak{M}(\sigma, F) \leq \ln \mu(\sigma+\delta, G_c) + \ln \sigma + O(1) \leq \Gamma(\sigma+\delta) + \ln \sigma + O(1), \quad \sigma \to +\infty.$$

This and (23) imply (21). □

Theorem 4. *Let $\Psi \in \Omega_{+\infty}$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. Then the following conditions are equivalent:*

(i) $\exists\, \varepsilon > 0\ \exists\, \sigma_1 \in \mathbb{R}\ \forall\, \sigma \geq \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma+\varepsilon) + \varepsilon\sigma$;

(ii) $\exists\, \delta > 0\ \exists\, \sigma_2 \in \mathbb{R}\ \forall\, \sigma \geq \sigma_2$: $\ln \mu(\sigma, G_c) \leq \Psi(\sigma+\delta) + \delta\sigma$;

(iii) $\exists\, \delta > 0\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln |b_n| \leq -\widetilde{\Psi}(\lambda_n - \delta) + \delta(\lambda_n - \delta)$.

Proof. The equivalence of conditions *(i)* and *(ii)* follows from Theorem 3. Furthermore, if $\delta \in \mathbb{R}$ and $\beta(\sigma) = \Psi(\sigma+\delta) + \delta\sigma$ for all $\sigma \in \mathbb{R}$, then by Lemma 9 we have $\widetilde{\beta}(x) = \widetilde{\Psi}(x-\delta) - \delta(x-\delta)$ for all $\in \mathbb{R}$, and therefore the equivalence of conditions *(ii)* and *(iii)* follows from Lemma 7. □

The following theorem can be proved analogously to Theorem 4.

Theorem 5. *Let $\Psi \in \Omega_{+\infty}$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. Then the following conditions are equivalent:*

(i) $\forall\, \varepsilon > 0\ \exists\, \sigma_1 \in \mathbb{R}\ \forall\, \sigma \geq \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma+\varepsilon) + \varepsilon\sigma$;

(ii) $\forall\, \delta > 0\ \exists\, \sigma_2 \in \mathbb{R}\ \forall\, \sigma \geq \sigma_2$: $\ln \mu(\sigma, G_c) \leq \Psi(\sigma+\delta) + \delta\sigma$;

(iii) $\forall\, \delta > 0\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln |b_n| \leq -\widetilde{\Psi}(\lambda_n - \delta) + \delta(\lambda_n - \delta)$.

Theorem 6. *Let $\Psi \in \Omega_{+\infty}$, Γ be the second Young conjugate function of Ψ, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If* (20) *holds, then for each $\varepsilon > 0$ we have*

$$\ln \mathfrak{M}(\sigma, F) - \Gamma\Big(\sigma + \varepsilon + \frac{\ln \sigma}{\Gamma'_+(\sigma+\varepsilon)}\Big) \to -\infty, \quad \sigma \to +\infty. \tag{24}$$

Proof. For some $\sigma_2 \in D_\Psi$ we have

$$\Gamma\Big(\sigma+\varepsilon+\frac{\ln\sigma}{\Gamma'_+(\sigma+\varepsilon)}\Big)-\Gamma(\sigma+\varepsilon)\geq \ln\sigma, \quad \sigma\geq\sigma_2. \tag{25}$$

In addition, by Lemma 7 condition (20) is equivalent to condition (22), and therefore by Theorem 3 we have $\ln\mathfrak{M}(\sigma,F)-\Gamma(\sigma+\varepsilon)-\ln\sigma\to-\infty$ as $\sigma\to+\infty$. This and (25) imply (24). □

The following theorem is a direct consequence of Theorem 6.

Theorem 7. *Let $\Psi\in\Omega_{+\infty}$, $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c>0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If* (20) *holds, then for each $\delta>0$ we have $\ln\mathfrak{M}(\sigma,F)-\Psi(\sigma+\delta\ln\sigma)\to-\infty$ as $\sigma\to+\infty$.*

Let $\Psi\in X$, $p>0$, and $\beta(\sigma)=\Psi(p\sigma)$ for all $\sigma\in\mathbb{R}$. Then $\widetilde{\beta}(x)=\widetilde{\Psi}(x/p)$ for all $x\in\mathbb{R}$ by Lemma 9. Taking this into account, and also using Theorem 7 and Lemma 7, it is easy to prove the following two theorems (they can also be derived from Theorem 4).

Theorem 8. *Let $\Psi\in\Omega_{+\infty}$, $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c>0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. Then the following conditions are equivalent:*

(i) $\exists\, q>1\ \exists\,\sigma_1\in\mathbb{R}\ \forall\,\sigma\geq\sigma_1$: $\ln\mathfrak{M}(\sigma,F)\leq\Psi(q\sigma)$;

(ii) $\exists\, p>1\ \exists\,\sigma_2\in\mathbb{R}\ \forall\,\sigma\geq\sigma_2$: $\ln\mu(\sigma,G_c)\leq\Psi(p\sigma)$;

(iii) $\exists\, p>1\ \exists\, n_0\in\mathbb{N}_0\ \forall\, n\geq n_0$: $\ln|b_n|\leq-\widetilde{\Psi}(\lambda_n/p)$.

Theorem 9. *Let $\Psi\in\Omega_{+\infty}$, $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c>0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. Then the following conditions are equivalent:*

(i) $\forall\, q>1\ \exists\,\sigma_1\in\mathbb{R}\ \forall\,\sigma\geq\sigma_1$: $\ln\mathfrak{M}(\sigma,F)\leq\Psi(q\sigma)$;

(ii) $\forall\, p>1\ \exists\,\sigma_2\in\mathbb{R}\ \forall\,\sigma\geq\sigma_2$: $\ln\mu(\sigma,G_c)\leq\Psi(p\sigma)$;

(iii) $\forall\, p>1\ \exists\, n_0\in\mathbb{N}_0\ \forall\, n\geq n_0$: $\ln|b_n|\leq-\widetilde{\Psi}(\lambda_n/p)$.

Theorem 10. *Let $\Psi\in\Omega_{+\infty}$, $F\in\mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c>0$, G_c be the series defined by* (12)*,* (11)*, and* (10)*, and let*

$$\delta_0=\varlimsup_{\sigma\to+\infty}\sigma\ln\sigma/\Psi(\sigma). \tag{26}$$

If $\delta_0<+\infty$ and (20) *holds, then for every $\delta>0$ we have*

$$\ln\mathfrak{M}(\sigma,F)-\Psi(\sigma+\delta_0+\delta)\to-\infty, \quad \sigma\to+\infty.$$

Proof. Let $\delta>0$ be a fixed number, and Γ be the second Young conjugate function of Ψ. Using Lemma 2, we see that in (26) we can replace Ψ with Γ. Then, using the convexity of the function Γ on $\mathbb{R}$, it is easy to prove that $\delta_0=\varlimsup_{\sigma\to+\infty}\ln\sigma/\Gamma'_+(\sigma)$. It remains to apply Theorem 6 with some fixed $\varepsilon\in(0,\delta)$. □

If $\Psi\in X$, $\delta\in\mathbb{R}$, and $\beta(\sigma)=\Psi(\sigma+\delta)$ for all $\sigma\in\mathbb{R}$, then $\widetilde{\beta}(x)=\widetilde{\Psi}(x)-\delta x$ for all $x\in\mathbb{R}$ by Lemma 9. Taking this into account, as well as Theorem 10 and Lemma 7, we obtain the following two theorems.

Theorem 11. *Let $\Psi \in \Omega_{+\infty}$, δ_0 be the quantity defined by* (26)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If $\delta_0 < +\infty$, then the following conditions are equivalent:*

(i) $\exists\, \varepsilon > 0\, \exists\, \sigma_1 \in \mathbb{R}\, \forall\, \sigma \geq \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma + \varepsilon)$;

(ii) $\exists\, \delta > 0\, \exists\, \sigma_2 \in \mathbb{R}\, \forall\, \sigma \geq \sigma_2$: $\ln \mu(\sigma, G_c) \leq \Psi(\sigma + \delta)$;

(iii) $\exists\, \delta > 0\, \exists\, n_0 \in \mathbb{N}_0\, \forall\, n \geq n_0$: $\ln |b_n| \leq -\widetilde{\Psi}(\lambda_n) + \delta\lambda_n$.

Theorem 12. *Let $\Psi \in \Omega_{+\infty}$, δ_0 be the quantity defined by* (26)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If $\delta_0 = 0$, then the following conditions are equivalent:*

(i) $\forall\, \varepsilon > 0\, \exists\, \sigma_1 \in \mathbb{R}\, \forall\, \sigma \geq \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma + \varepsilon)$;

(ii) $\forall\, \delta > 0\, \exists\, \sigma_2 \in \mathbb{R}\, \forall\, \sigma \geq \sigma_2$: $\ln \mu(\sigma, G_c) \leq \Psi(\sigma + \delta)$;

(iii) $\forall\, \delta > 0\, \exists\, n_0 \in \mathbb{N}_0\, \forall\, n \geq n_0$: $\ln |b_n| \leq -\widetilde{\Psi}(\lambda_n) + \delta\lambda_n$.

Theorem 13. *Let $\Psi \in \Omega_{+\infty}$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, h be a nondecreasing, continuous, unbounded from above function in some neighborhood of the point $+\infty$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (11)*, and* (10)*. If*

$$\exists\, x_0 \in \mathbb{R}\, ; \forall\, x \geq x_0 : \; \ln x \leq h(\Psi(\psi(x))) \tag{27}$$

and (20) *holds, then for every $\delta > 0$ we have*

$$\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma) + h(\Psi(\sigma) + \delta) + \ln \sigma + \delta - \ln \delta + o(1), \quad \sigma \to +\infty. \tag{28}$$

Proof. Let Γ be the second Young conjugate function of Ψ. By Lemma 7, condition (20) is equivalent to condition (22). Using Lemma 5, it is easy to prove that condition (27) is equivalent to the condition

$$\exists\, \sigma_2 \in D_\Psi\; \forall\, \sigma \geq \sigma_2 : \; \ln \Gamma'_+(\sigma) \leq h(\Gamma(\sigma)). \tag{29}$$

Let $\delta > 0$ be a fixed number. We also fix $\sigma_3 \in D_\Psi$ such that $\Gamma'_+(\sigma_3) > 0$ and for each $\sigma \geq \sigma_3$ we denote by $\varepsilon(\sigma)$ that positive value of ε for which $\varepsilon\Gamma'_-(\sigma + \varepsilon) \leq \delta \leq \varepsilon\Gamma'_+(\sigma + \varepsilon)$. Note that

$$\Gamma(\sigma + \varepsilon(\sigma)) - \Gamma(\sigma) \leq \varepsilon(\sigma)\Gamma'_-(\sigma + \varepsilon(\sigma)) \leq \delta, \quad \sigma \geq \sigma_3. \tag{30}$$

Using Theorem 1 with $\varepsilon = \varepsilon(\sigma)$ and taking into account (22), (30), and (29), we have

$$\begin{aligned}
\ln \mathfrak{M}(\sigma, F) &\leq \Gamma(\sigma + \varepsilon(\sigma)) + \ln \frac{1 + \varepsilon(\sigma)c + \sigma c}{\varepsilon(\sigma)c} \\
&= \Gamma(\sigma + \varepsilon(\sigma)) + \ln \frac{\sigma}{\varepsilon(\sigma)} + o(1) \\
&= \Gamma(\sigma + \varepsilon(\sigma)) + \ln \sigma + \ln \Gamma'_+(\sigma + \varepsilon(\sigma)) - \ln \delta + o(1) \\
&\leq \Gamma(\sigma) + \delta + \ln \sigma + h(\Gamma(\sigma) + \delta) - \ln \delta + o(1)
\end{aligned}$$

as $\sigma \to +\infty$. This implies (28). □

The following theorem is a direct consequence of Theorem 13.

Theorem 14. *Let $\Psi \in \Omega_{+\infty}$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_{+\infty}$ be the Dirichlet series of the form* (1)*, $c > 0$, G_c be the series defined by* (12), (11), *and* (10)*, and*

$$\Delta = \varlimsup_{x\to+\infty} \ln x/\Psi(\psi(x)). \tag{31}$$

If $\Delta < +\infty$ and (20) *holds, then $\ln \mathfrak{M}(\sigma, F) \le (1 + \Delta + o(1))\Psi(\sigma)$ as $\sigma \to +\infty$.*

If $\Psi \in X$, $p > 0$, and $\beta(\sigma) = p\Psi(\sigma)$ for all $\sigma \in \mathbb{R}$, then $\widetilde{\beta}(x) = p\widetilde{\Psi}(x/p)$ for all $x \in \mathbb{R}$ by Lemma 9. Therefore, from Theorem 14 and Lemma 7 we obtain the following two theorems.

Theorem 15. *Let $\Psi \in \Omega_{+\infty}$, Δ be the quantity defined by* (31)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12), (11), *and* (10). *If $\Delta < +\infty$, then the following conditions are equivalent:*

(i) $\exists\, q > 0\ \exists\, \sigma_1 \in \mathbb{R}\ \forall\, \sigma \ge \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \le q\Psi(\sigma)$;

(ii) $\exists\, p > 0\ \exists\, \sigma_2 \in \mathbb{R}\ \forall\, \sigma \ge \sigma_2$: $\ln \mu(\sigma, G_c) \le p\Psi(\sigma)$;

(iii) $\exists\, p > 0\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \ge n_0$: $\ln |b_n| \le -p\widetilde{\Psi}(\lambda_n/p)$.

Theorem 16. *Let $\Psi \in \Omega_{+\infty}$, Δ be the quantity defined by* (31)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12), (11), *and* (10). *If $\Delta = 0$, then the following conditions are equivalent:*

(i) $\forall\, q > 1\ \exists\, \sigma_1 \in \mathbb{R}\ \forall\, \sigma \ge \sigma_1$: $\ln \mathfrak{M}(\sigma, F) \le q\Psi(\sigma)$;

(ii) $\forall\, p > 1\ \exists\, \sigma_2 \in \mathbb{R}\ \forall\, \sigma \ge \sigma_2$: $\ln \mu(\sigma, G_c) \le p\Psi(\sigma)$;

(iii) $\forall\, p > 1\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \ge n_0$: $\ln |b_n| \le -p\widetilde{\Psi}(\lambda_n/p)$.

Let us now turn to the case of Dirichlet series absolutely converging in a half-plane.

Consider the function $y : [0, +\infty] \to [0, 1]$, defined as follows:

$$y(q) = \frac{\sqrt{q+1} - 1}{\sqrt{q+1} + 1}, \quad q \in [0, +\infty]; \tag{32}$$

here, of course, $y(+\infty) = 1$. Note that this function is continuous, increasing on $[0, +\infty]$, and the interval $[0, 1]$ is its range.

Theorem 17. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, F_2 be the series defined by* (5) *and* (4)*, and $q = \varliminf_{x\to+\infty} |\psi(x)|x$. If $q > 0$ and*

$$\exists\, \sigma_1 \in D_\Psi\ \forall\, \sigma \in [\sigma_1, 0): \ \ln \mu(\sigma, F_2) \le \Psi(\sigma), \tag{33}$$

then for every $\eta \in (0, y(q))$, where $y(q)$ is defined by (32)*, there exists $\sigma_2 \in D_\Psi$ such that*

$$\ln \mathfrak{M}(\sigma, F) \le \Psi(\eta\sigma), \quad \sigma \in [\sigma_2, 0). \tag{34}$$

Proof. Let Γ be the second Young conjugate function of Ψ. Using Lemma 5, it is easy to prove that $q = \underline{\lim}_{\sigma\uparrow 0} |\sigma|\Gamma'_+(\sigma)$. From properties of the function y it follows that there exists a unique number $p < q$ for which $\eta = y(p)$. Then there exists $\sigma_3 \in D_\Psi$ such that $|\sigma|\Gamma'_+(\sigma) \geq p$ for all $\sigma \in [\sigma_3, 0)$. Setting $\delta = (\sqrt{p+1} - 1)/\sqrt{p+1}$, we get

$$\Gamma(\eta\sigma) - \Gamma(\delta\sigma) \geq (\delta - \eta)|\sigma|\Gamma'_+(\delta\sigma) \geq \frac{\delta - \eta}{\delta} p = \frac{\delta}{1-\delta}, \quad \sigma \in [\sigma_3, 0). \tag{35}$$

Next, we note that condition (33), according to Lemma 7, is equivalent to the condition

$$\exists\, \sigma_4 \in D_\Psi\ \forall\, \sigma \in [\sigma_4, 0) :\ \ln \mu(\sigma, F_2) \leq \Gamma(\sigma). \tag{36}$$

Setting $\sigma_2 = \max\{\sigma_3, \sigma_4\}$ and using Theorem E, (36), and (35), for all $\sigma \in [\sigma_2, 0)$ we have

$$\ln \mathfrak{M}(\sigma, F) \leq \ln \mu(\delta\sigma, F_2) + \ln \frac{1}{1-\delta} \leq \Gamma(\delta\sigma) + \frac{\delta}{1-\delta} \leq \Gamma(\eta\sigma) \leq \Psi(\eta\sigma),$$

and therefore, the theorem is proved. □

Using Theorem 17 and Lemmas 9 and 7, we obtain the following two results.

Theorem 18. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, and F_2 be the series defined by* (5) *and* (4)*. If $\underline{\lim}_{x\to+\infty} |\psi(x)|x > 0$, then the following conditions are equivalent:*

(i) $\exists\, \delta \in (0,1)\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\delta\sigma)$;

(ii) $\exists\, \eta \in (0,1)\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln \mu(\sigma, F_2) \leq \Psi(\eta\sigma)$;

(iii) $\exists\, \eta \in (0,1)\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln |S_n| \leq -\widetilde{\Psi}(\lambda_n/\eta)$.

Theorem 19. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, and F_2 be the series defined by* (5) *and* (4)*. If $|\psi(x)|x \to +\infty$ as $x \to +\infty$, then the following conditions are equivalent:*

(i) $\forall\, \delta \in (0,1)\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln \mathfrak{M}(\sigma, F) \leq \Psi(\delta\sigma)$;

(ii) $\forall\, \eta \in (0,1)\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln \mu(\sigma, F_2) \leq \Psi(\eta\sigma)$;

(iii) $\forall\, \eta \in (0,1)\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln |S_n| \leq -\widetilde{\Psi}(\lambda_n/\eta)$.

Theorem 20. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, G_c be the series defined by* (12), (16), *and* (10)*, and $q = \underline{\lim}_{x\to+\infty} |\psi(x)|x/\ln(1/|\psi(x)|)$. If $q > 1$ and the condition*

$$\exists\, \sigma_1 \in D_\Psi\ \forall\, \sigma \in [\sigma_1, 0) :\ \ln \mu(\sigma, G_c) \leq \Psi(\sigma) \tag{37}$$

holds, then for every positive $\eta < (q-1)/q$ there exists $\sigma_2 \in D_\Psi$ such that we have (34).

Proof. Let Γ be the second Young conjugate function of Ψ. Using Lemma 5, it is easy to prove that $q = \underline{\lim}_{\sigma\uparrow 0} |\sigma|\Gamma'_+(\sigma)/\ln(1/|\sigma|)$. From the condition $q > 1$ and the inequality $\eta < (q-1)/q$ it follows that there exist numbers $\delta \in (\eta, 1)$ and $p \in (1, q)$ such that

$\eta/\delta < (p-1)/p$. Let us fix some $r \in (p,q)$. Then there exists $\sigma_3 \in D_\Psi$ such that $\sigma_3 > -1$ and $|\sigma|\Gamma'_+(\sigma) \geq r\ln(1/|\sigma|)$ for all $\sigma \in [\sigma_3, 0)$. Therefore,

$$\Gamma(\eta\sigma) - \Gamma(\delta\sigma) \geq (\delta-\eta)|\sigma|\Gamma'_+(\delta\sigma) \geq \frac{\delta-\eta}{\delta} r \ln\frac{1}{\delta|\sigma|} \geq \frac{r}{p}\ln\frac{1}{|\sigma|}, \quad \sigma \in [\sigma_3, 0). \tag{38}$$

Furthermore, we note that by Lemma 7, condition (37) is equivalent to the condition

$$\exists\, \sigma_4 \in D_\Psi\ \forall\, \sigma \in [\sigma_4, 0):\ \ln\mu(\sigma, G_c) \leq \Gamma(\sigma). \tag{39}$$

Using Theorem 2 (see Remark 2), condition (39), the inequality $r/p > 1$, and (38), we obtain

$$\ln\mathfrak{M}(\sigma, F) \leq \ln\mu(\delta\sigma, G_c) + \ln\frac{1}{|\sigma|} + O(1) \leq \Gamma(\delta\sigma) + \frac{r}{p}\ln\frac{1}{|\sigma|} \leq \Gamma(\eta\sigma) \leq \Psi(\eta\sigma)$$

as $\sigma \uparrow 0$, that is, (34) holds for some $\sigma_2 \in D_\Psi$. The theorem is proved. □

Using Theorem 20 and Lemmas 9 and 7, we obtain the following two theorems.

Theorem 21. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (16)*, and* (10)*. If $\varliminf_{x\to+\infty} |\psi(x)|x/\ln(1/|\psi(x)|) > 1$, then the following conditions are equivalent:*

(i) $\exists\, \delta \in (0,1)\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln\mathfrak{M}(\sigma, F) \leq \Psi(\delta\sigma)$;

(ii) $\exists\, \eta \in (0,1)\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln\mu(\sigma, G_c) \leq \Psi(\eta\sigma)$;

(iii) $\exists\, \eta \in (0,1)\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln|b_n| \leq -\widetilde{\Psi}(\lambda_n/\eta)$.

Theorem 22. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (16)*, and* (10)*. If $|\psi(x)|x/\ln(1/|\psi(x)| \to +\infty$ as $x \to +\infty$, then the following conditions are equivalent:*

(i) $\forall\, \delta \in (0,1)\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln\mathfrak{M}(\sigma, F) \leq \Psi(\delta\sigma)$;

(ii) $\forall\, \eta \in (0,1)\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln\mu(\sigma, G_c) \leq \Psi(\eta\sigma)$;

(iii) $\forall\, \eta \in (0,1)\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \geq n_0$: $\ln|b_n| \leq -\widetilde{\Psi}(\lambda_n/\eta)$.

Theorem 23. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, h be a nondecreasing, continuous, unbounded from above function in some neighborhood of the point $+\infty$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, and F_2 be the series defined by* (5) *and* (4)*. If conditions* (27) *and* (33) *hold, then there exists $\sigma_2 \in D_\Psi$ such that for each $\eta > 0$ and all $\sigma \in [\sigma_2, 0)$ we have*

$$\ln\mathfrak{M}(\sigma, F) \leq \Psi(\sigma) + \eta|\sigma| + h(\Psi(\sigma + \eta|\sigma|) - \ln\eta. \tag{40}$$

Proof. Let Γ be the second Young conjugate function of Ψ. By Lemma 7, condition (33) is equivalent to condition (36). Using Lemma 5, it is easy to prove that condition (27) is equivalent to the condition

$$\exists\, \sigma_3 \in D_\Psi\ \forall\, \sigma \in [\sigma_3, 0):\ \ln\Gamma'_+(\sigma) \leq h(\Gamma(\sigma)). \tag{41}$$

Since $\Gamma \in \Omega_0$ and Γ are convex on D_Ψ, then $\Gamma'_+(\sigma) \nearrow +\infty$ as $\sigma \uparrow 0$. We choose $\sigma_2 < 0$ such that the inequalities $\Gamma'_-(\sigma_2) > 0$, $\sigma_2 \geq \sigma_3$, and $\sigma_2 \geq \sigma_4$ hold, where σ_4 is the number from condition (36). We fix arbitrary $\eta > 0$ and $\sigma \in [\sigma_2, 0)$. Consider the functions $y_1 = 1 - \delta$ and $y_2 = \eta/\Gamma'_+(\delta\sigma)$ of the variable δ, defined on the interval $(0, 1]$. On this interval the function y_1 is continuous, decreasing, and takes all values from $[0, 1)$, and the function y_2 is nondecreasing with $y_2(0+0) = 0$ and $y_2(1) = \eta/\Gamma'_+(\sigma) > 0$. So, as is easy to see, on the interval $(0, 1)$ there exists a unique number $\delta = \delta(\sigma)$ such that $\eta/\Gamma'_+(\delta\sigma) \leq 1 - \delta \leq \eta/\Gamma'_-(\delta\sigma)$. Note that

$$\Gamma(\delta\sigma) - \Gamma(\sigma) \leq (1-\delta)|\sigma|\Gamma'_-(\delta\sigma) \leq \eta|\sigma|. \tag{42}$$

Using Theorem E and taking into account (36), (42) and (41), we have

$$\begin{aligned}\ln \mathfrak{M}(\sigma, F) &\leq \Gamma(\delta\sigma) - \ln(1-\delta) \\ &\leq \Gamma(\sigma) + \eta|\sigma| - \ln \eta + \ln \Gamma'_+(\delta\sigma) \\ &\leq \Gamma(\sigma) + \eta|\sigma| - \ln \eta + h(\Gamma(\sigma + \eta|\sigma|).\end{aligned}$$

This implies (40). □

Theorem 24. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, h be a nondecreasing, continuous, unbounded from above function in some neighborhood of the point $+\infty$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12), (16)*, and* (10)*. If conditions* (27) *and* (37) *hold, then for each $\varepsilon > 0$ we have*

$$\ln \mathfrak{M}(\sigma, F) \leq \Psi(\sigma) + h(\Psi(\sigma) + \varepsilon) + \varepsilon - \ln \varepsilon - \ln c + o(1), \quad \sigma \uparrow 0. \tag{43}$$

Proof. Let Γ be the second Young conjugate function of Ψ. By Lemma 7, condition (37) is equivalent to condition (39), and by Lemma 5, condition (27) is equivalent to condition (41). Let also σ_3 and σ_4 be the numbers from conditions (41) and (39), respectively, and $\varepsilon > 0$ be a fixed number.

We choose $\sigma_2 < 0$ such that the inequalities $\Gamma'_-(\sigma_2) > 0$, $\sigma_2 \geq \sigma_3$, and $\sigma_2 \geq \sigma_4$ hold. Let $\sigma \in [\sigma_2, 0)$. As is easy to see, on the interval $(0, 1)$ there exists a unique number $\delta = \delta(\sigma)$ such that $(1-\delta)|\sigma|\Gamma'_-(\delta\sigma) \leq \varepsilon \leq (1-\delta)|\sigma|\Gamma'_+(\delta\sigma)$. Note that

$$\Gamma(\delta\sigma) - \Gamma(\sigma) \leq (1-\delta)|\sigma|\Gamma'_-(\delta\sigma) \leq \varepsilon. \tag{44}$$

Using Theorem 2 (see Remark 2) and taking into account (39), (44), and (41), we have

$$\begin{aligned}\ln \mathfrak{M}(\sigma, F) &\leq \Gamma(\delta\sigma) + \ln\left(\frac{1}{1-\delta} + 1 + \frac{1}{(1-\delta)|\sigma|c}\right) \\ &= \Gamma(\delta\sigma) + \ln \frac{1}{(1-\delta)|\sigma|c} + o(1) \\ &= \Gamma(\delta\sigma) + \ln \Gamma'_+(\delta\sigma) - \ln \varepsilon - \ln c + o(1) \\ &\leq \Gamma(\sigma) + \varepsilon + h(\Gamma(\sigma) + \varepsilon) - \ln \varepsilon - \ln c + o(1)\end{aligned}$$

as $\sigma \uparrow 0$. This implies (43). □

The following theorem is a direct consequence of Theorem 24.

Theorem 25. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, Δ be the quantity defined by* (31)*, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (16)*, and* (10)*. If $\Delta < +\infty$ and condition* (20) *holds, then we have $\ln \mathfrak{M}(\sigma, F) \le (1 + \Delta + o(1))\Psi(\sigma)$ as $\sigma \uparrow 0$.*

Using Theorem 25 and Lemmas 9 and 7, we obtain the following two statements.

Theorem 26. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, Δ be the quantity defined by* (31)*, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (16)*, and* (10)*. If $\Delta < +\infty$, then the following conditions are equivalent:*

(i) $\exists\, q > 0\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln \mathfrak{M}(\sigma, F) \le q\Psi(\sigma)$;

(ii) $\exists\, p > 0\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln \mu(\sigma, G_c) \le p\Psi(\sigma)$;

(iii) $\exists\, p > 0\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \ge n_0$: $\ln |b_n| \le -p\widetilde{\Psi}(\lambda_n/p)$.

Theorem 27. *Let $\Psi \in \Omega_0$, $\psi(x)$ be the right-hand derivative of $\widetilde{\Psi}(x)$ at each point $x \in \mathbb{R}$, Δ be the quantity defined by* (31)*, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and G_c be the series defined by* (12)*,* (16)*, and* (10)*. If $\Delta = 0$, then the following conditions are equivalent:*

(i) $\forall\, q > 1\ \exists\, \sigma_1 < 0\ \forall\, \sigma \in [\sigma_1, 0)$: $\ln \mathfrak{M}(\sigma, F) \le q\Psi(\sigma)$;

(ii) $\forall\, p > 1\ \exists\, \sigma_2 < 0\ \forall\, \sigma \in [\sigma_2, 0)$: $\ln \mu(\sigma, G_c) \le p\Psi(\sigma)$;

(iii) $\forall\, p > 1\ \exists\, n_0 \in \mathbb{N}_0\ \forall\, n \ge n_0$: $\ln |b_n| \le -p\widetilde{\Psi}(\lambda_n/p)$.

Note that estimates of the type obtained above are often used to study the growth of analytic functions (see, for example, [5, 8, 9, 21, 24, 25]). In particular, such estimates are necessary to describe the minimal growth of analytic functions with given zeros (see [1–4, 7, 20, 39]).

5 Global estimates for sums of Dirichlet series

Conditions, under which some global estimates for the sum of a series $F \in \mathcal{D}_A$ hold, are found in the works [23, 38]. Here we supplement the results from [23, 38], and also investigate other global estimates.

By Ω' we denote the class of all continuously differentiable, positive on $\mathbb{R}$ functions Φ such that Φ' is an increasing, positive on $\mathbb{R}$ function. Let Ω be the class of all continuous, positive, increasing on $\mathbb{R}$ functions Φ such that $\Phi(\sigma)/\sigma \to +\infty$ as $\sigma \to +\infty$. It is clear that $\Omega' \subset \Omega \subset X$.

Let $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form (1). M.M. Sheremeta [38], in the case when $\Psi \in \Omega'$, indicated a condition on the sequence $(|a_n|)_{n\in\mathbb{N}_0}$ (in terms of the sequence $(R_n)_{n\in\mathbb{N}_0}$ defined by (2)), which is necessary and sufficient in order that for every $q > 1$ there exists a constant $B \in \mathbb{R}$ such that

$$\ln \mathfrak{M}(\sigma, F) \le \Psi(q\sigma) + B, \quad \sigma \in \mathbb{R}. \tag{45}$$

In [23], this result is extended to the case when $\Phi \in \Omega$. In the general case, that is, for an arbitrary function $\Phi \in X$, we have the following theorem.

Theorem 28. *Let $\Psi \in X$, $q_0 \geq 0$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1), *$c > 0$, and $(b_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (11) *and* (10). *Then for every $q > q_0$ there exists a constant $B \in \mathbb{R}$ such that* (45) *holds if and only if for every $p > q_0$ there exists a constant $C \in \mathbb{R}$ such that*

$$\ln b_n \leq -\widetilde{\Psi}(\lambda_n/p) + C, \quad n \in \mathbb{N}_0. \tag{46}$$

Proof. Let N be the number defined by (10), and G_c be the series defined by (12). Note that the functions $\mu(\sigma, F)$ and $\mu(\sigma, G_c)$ are continuous and positive on $\mathbb{R}$, and for all σ from some neighborhood of the point $-\infty$ we have $\mu(\sigma, F) = |a_N|e^{\sigma\lambda_N}$ and $\mu(\sigma, G_c) = b_N e^{\sigma\lambda_N}$. It is also clear that $\mathfrak{M}(\sigma, F) \sim |a_N|e^{\sigma\lambda_N}$ as $\sigma \to -\infty$. Let us fix some number $\sigma_0 > 0$. From what has been said it immediately follows that there exist constants $c_1 \in (0, 1)$ and $c_2 > 1$ such that $c_1\mu(\sigma, G_c) \leq \mathfrak{M}(\sigma, F) \leq c_2\mu(\sigma, G_c)$ for all $\sigma \leq \sigma_0$.

Sufficiency. Suppose that for every $p > q_0$ there exists a real constant $C = C(p)$ such that (46) holds, and let $q > q_0$ be a fixed number. We put $p_1 = q$ and fix some $p_2 \in (q_0, q)$. If $j \in \{1, 2\}$, then $\ln b_n \leq -\widetilde{\Psi}(\lambda_n/p_j) + C_j$ for all $n \in \mathbb{N}_0$, where $C_j = C(p_j)$, and therefore by Lemmas 8 and 9 we have

$$\ln \mu(\sigma, G_c) \leq \Psi(p_j\sigma) + C_j, \quad \sigma \in \mathbb{R}. \tag{47}$$

Since $\mathfrak{M}(\sigma, F) \leq c_2\mu(\sigma, G_c)$ for all $\sigma \leq \sigma_0$, then, using (47) with $j = 1$, we obtain

$$\ln \mathfrak{M}(\sigma, F) \leq \Psi(q\sigma) + C_1 + \ln c_2, \quad \sigma \leq \sigma_0. \tag{48}$$

Next, for each $\sigma \geq \sigma_0$ we set $\varepsilon(\sigma) = (q - p_2)\sigma/p_2$ and let $\varepsilon_0 = \varepsilon(\sigma_0)$. Then by Theorem 1 for all $\sigma \geq \sigma_0$ we have

$$\mathfrak{M}(\sigma, F) \leq \mu(\sigma + \varepsilon(\sigma), G_c)\Big(\frac{e^{\varepsilon_0 c}}{e^{\varepsilon_0 c} - 1} + \frac{\sigma}{\varepsilon(\sigma)}\Big) = c_3\mu\Big(\frac{q\sigma}{p_2}, G_c\Big), \quad c_3 := \frac{e^{\varepsilon_0 c}}{e^{\varepsilon_0 c} - 1} + \frac{p_2}{q - p_2}.$$

So, using (47) with $j = 2$, we get

$$\ln \mathfrak{M}(\sigma, F) \leq \Psi(q\sigma) + C_2 + \ln c_3, \quad \sigma \geq \sigma_0. \tag{49}$$

Taking $B = \max\{C_1 + \ln c_2, C_2 + \ln c_3\}$, from (48) and (49) we see that (45) holds.

Necessity. Suppose that for every $q > q_0$ there exists a real constant $B = B(q)$ such that (45) holds. Let $p > q_0$ be a fixed number and $B_1 = B(p)$, i.e. for all $\sigma \in \mathbb{R}$ we have $\ln \mathfrak{M}(\sigma, F) \leq \Psi(p\sigma) + B_1$. Let also c_1 be the constant defined above. By the definition of the constant c_1 and by Theorem 1 for all $\sigma \in \mathbb{R}$ we have $c_1\mu(\sigma, G_c) \leq \mathfrak{M}(\sigma, F)$, and therefore

$$\ln \mu(\sigma, G_c) \leq \ln \mathfrak{M}(\sigma, F) - \ln c_1 \leq \Psi(p\sigma) + B_1 - \ln c_1, \quad \sigma \in \mathbb{R}.$$

Then, taking $C = B_1 - \ln c_1$, by Lemmas 8 and 9 we obtain (46). □

Let $\Psi \in \Omega'$, and $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form (1). M.M. Sheremeta [38] also considered the following problem posed by B.V. Vynnyts'kyi: find a condition on the sequence $(|a_n|)_{n\in\mathbb{N}_0}$ that is necessary and sufficient in order that there exist positive constants q and B such that

$$\ln \mathfrak{M}(\sigma, F) \leq q\Psi(\sigma + B), \quad \sigma \in \mathbb{R}. \tag{50}$$

The result in [38], which gave such a condition (in terms of the sequence $(R_n)_{n\in\mathbb{N}_0}$ defined by (2)), contained a minor inaccuracy, which was corrected in [23]. In addition, in [23], a solution to the formulated problem was obtained for the case when $\Psi \in \Omega$. In the general case, that is, in the case of arbitrary $\Psi \in X$, this problem is open. However, we succeeded in proving the following theorem.

Theorem 29. *Let $\Psi \in X$, $q_0 \geq 0$, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, N be the number defined by* (10)*, $0 < c < \lambda_{N+1} - \lambda_N$, and $(b_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (11)*. If $\lambda_N > 0$, then for every $q > q_0$ there exists a constant $B \in \mathbb{R}$ such that* (50) *holds if and only if for every $p > q_0$ there exists a constant $C \in \mathbb{R}$ such that*

$$\ln b_n \leq -p\widetilde{\Psi}(\lambda_n/p) + C\lambda_n, \quad n \in \mathbb{N}_0. \tag{51}$$

Proof. Let G_c be the series defined by (12). Since $0 < c < \lambda_{N+1} - \lambda_N$, then $b_N = |a_N|$, and therefore there exists $\sigma_1 < 0$ such that $\mu(\sigma, G_c) = \mu(\sigma, F) = |a_N|e^{\sigma\lambda_N} \leq \mathfrak{M}(\sigma, F)$ for all $\sigma \leq \sigma_1$. Furthermore, since $\lambda_N > 0$ and $\mathfrak{M}(\sigma, F) \sim |a_N|e^{\sigma\lambda_N}$ as $\sigma \to -\infty$, there exists $\sigma_2 < 0$ such that $\mathfrak{M}(\sigma, F) \leq |a_N|e^{(\sigma+1)\lambda_N} \leq \mu(\sigma + 1, G_c)$ for all $\sigma \leq \sigma_2$.

Sufficiency. Suppose that for every $p > q_0$ there exists a real constant $C = C(p)$ such that (51) holds, and let $q > q_0$ be a fixed number. We put $p_1 = q$ and fix some $p_2 \in (q_0, q)$. If $j \in \{1, 2\}$, then $\ln b_n \leq -p_j\widetilde{\Psi}(\lambda_n/p_j) + C_j\lambda_n$ for all $n \in \mathbb{N}_0$, where $C_j = C(p_j)$, and therefore by Lemmas 8 and 9 we have

$$\ln \mu(\sigma, G_c) \leq p_j\Psi(\sigma + C_j), \quad \sigma \in \mathbb{R}. \tag{52}$$

Using Theorem 1, we obtain $\ln \mathfrak{M}(\sigma, F) \leq \ln \mu(\sigma + 1, G_c) + 2\ln\sigma$ for all sufficiently large $\sigma \geq 0$. Since $\ln \mu(\sigma, G_c) \geq \ln|a_N| + \sigma\lambda_N$ for all $\sigma \in \mathbb{R}$, there exists a number $\sigma_3 \geq 0$ such that $\ln \mathfrak{M}(\sigma, F) \leq q \ln \mu(\sigma + 1, G_c)/p_2$ for all $\sigma \geq \sigma_3$. Then for all $\sigma \in [\sigma_2, \sigma_3]$ we have

$$\ln \mathfrak{M}(\sigma, F) \leq \ln \mathfrak{M}(\sigma_3, F) \leq \frac{q}{p_2} \ln \mu(\sigma_3 + 1, G_c) \leq \frac{q}{p_2} \ln \mu(\sigma + \sigma_3 - \sigma_2 + 1, G_c).$$

Put $B = \max\{1 + C_1, \sigma_3 - \sigma_2 + 1 + C_2\}$. If $\sigma \leq \sigma_2$, then using the monotonicity of the function $\mu(\sigma, G_c)$, the inequality $1 \leq B - C_1$, and (52) with $j = 1$, we obtain

$$\ln \mathfrak{M}(\sigma, F) \leq \ln \mu(\sigma + 1, G_c) \leq \ln \mu(\sigma + B - C_1, G_c) \leq q\Psi(\sigma + B).$$

Similarly, using the inequalities $1 \leq \sigma_3 - \sigma_2 + 1 \leq B - C_2$ and (52) with $j = 2$, for all $\sigma \geq \sigma_2$ we have

$$\ln \mathfrak{M}(\sigma, F) \leq \frac{q}{p_2} \ln \mu(\sigma + \sigma_3 - \sigma_2 + 1, G_c) \leq \frac{q}{p_2} \ln \mu(\sigma + B - C_2, G_c) \leq q\Psi(\sigma + B).$$

Therefore, (50) holds.

Necessity. Suppose that for every $q > q_0$ there exists a real constant $B = B(q)$ such that (50) holds. Let $p > q_0$ be a fixed number and $B_1 = B(p)$, i.e. for all $\sigma \in \mathbb{R}$ we have $\ln \mathfrak{M}(\sigma, F) \leq p\Psi(\sigma + B_1)$. Let $C = -\sigma_1 + B_1$, where $\sigma_1 < 0$ is the number defined above. Using the definition of the number σ_1 and Theorem 1, as well as the inequality $0 < C - B_1$ and the monotonicity of the function $\mathfrak{M}(\sigma, F)$, for all $\sigma \notin [\sigma_1, 0]$ we have

$$\ln \mu(\sigma, G_c) \leq \ln \mathfrak{M}(\sigma, F) < \ln \mathfrak{M}(\sigma + C - B_1, F) \leq p\Psi(\sigma + C).$$

Similarly, for all $\sigma \in [\sigma_1, 0]$ we obtain

$$\begin{aligned}\ln \mu(\sigma, G_c) &\leq \ln \mu(0, G_c) \leq \ln \mathfrak{M}(0, F) \\ &\leq \ln \mathfrak{M}(\sigma - \sigma_1, F) = \ln \mathfrak{M}(\sigma + C - B_1, F) \leq p\Psi(\sigma + C).\end{aligned}$$

Therefore, $\ln \mu(\sigma, G_c) \leq p\Psi(\sigma + C)$ for all $\sigma \in \mathbb{R}$. Hence, by Lemmas 8 and 9, (51) holds. □

Remark 4. *A detailed analysis of the proof of the following theorem shows that in Theorem 29 the condition* $c < \lambda_{N+1} - \lambda_N$ *can be omitted, but then* (51) *must be replaced by*

$$\ln b_n \le -p\widetilde{\Psi}(\lambda_n/p) + \ln(|a_N|/b_N) + C\lambda_n, \quad n \in \mathbb{N}_0.$$

Theorem 30. *Let* $\Psi \in X$, δ_0 *be the quantity defined by* (26), $F \in \mathcal{D}_{+\infty}$ *be a Dirichlet series of the form* (1), N *be the number defined by* (10), $c > 0$, *and* $(b_n)_{n\in\mathbb{N}_0}$ *be the sequence defined by* (11). *If* $\lambda_N > 0$ *and* $\delta_0 < +\infty$, *then there exists a constant* $\varepsilon \in \mathbb{R}$ *such that*

$$\ln \mathfrak{M}(\sigma, F) \le \Psi(\sigma + \varepsilon), \quad \sigma \in \mathbb{R}, \tag{53}$$

if and only if there exists a constant $\delta \in \mathbb{R}$ *such that*

$$\ln b_n \le -\widetilde{\Psi}(\lambda_n) + \ln(|a_N|/b_N) + \delta\lambda_n, \quad n \in \mathbb{N}_0. \tag{54}$$

Proof. Let G_c be the series defined by (12).

Necessity. Suppose that for some constant $\varepsilon \in \mathbb{R}$ we have (53). Reasoning as in the proof of Theorem 28, for some $\sigma_1 < 0$ and all $\sigma \le \sigma_1$ we have the inequality $\frac{|a_N|}{b_N}\mu(\sigma, G_c) \le \mathfrak{M}(\sigma, F)$. By Theorem 1 this inequality also holds for all $\sigma \ge 0$. Therefore, for each $\sigma \in \mathbb{R}$ we have

$$\ln \mu(\sigma, G_c) \le \ln \mathfrak{M}(\sigma - \sigma_1, F) - \ln(|a_N|/b_N) \le \Psi(\sigma - \sigma_1 + \varepsilon) - \ln(|a_N|/b_N).$$

Taking $\delta = -\sigma_1 + \varepsilon$, by Lemmas 8 and 9 we obtain (54).

Sufficiency. Suppose that there exists a constant $\delta \in \mathbb{R}$ such that (54) holds. Then by Lemmas 8 and 9 we obtain

$$\ln \mu(\sigma, G_c) \le \Psi(\sigma + \delta) - \ln(|a_N|/b_N), \quad \sigma \in \mathbb{R}. \tag{55}$$

Reasoning as in the proof of Theorem 29, for some $\sigma_2 < 0$ and all $\sigma \le \sigma_2$ we have $\mathfrak{M}(\sigma, F) \le \frac{|a_N|}{b_N}\mu(\sigma + 1, G_c)$. Furthermore, according to the condition $\delta_0 < +\infty$, there exist constants $\sigma_3 > 0$ and $\eta > 0$ such that for all $\sigma \ge \sigma_3$ we have $\Psi(\sigma) \ge \eta\sigma \ln \sigma$.

Without loss of generality, we can assume that Ψ is nondecreasing, convex, and takes finite values on $\mathbb{R}$ (otherwise we set $\gamma(\sigma) = \ln \mu(\sigma - \delta, G_c) + \ln(|a_N|/b_N)$ for all $\sigma < \sigma_3$, $\gamma(\sigma) = \max\{\ln \mu(\sigma - \delta, G_c) + \ln(|a_N|/b_N), \eta\sigma \ln \sigma\}$ for all $\sigma \ge \sigma_3$, and, using Lemma 3, everywhere below instead of Ψ we consider the function $\widetilde{\widetilde{\gamma}}$).

Using the convexity of the function Ψ, it is easy to prove the existence of a number $\sigma_4 \ge \sigma_3$ such that $\Psi'_+(\sigma) \ge \eta \ln \sigma$ for all $\sigma \ge \sigma_4$. Let us fix a number ζ such that $(\zeta - 1 - \delta)\eta > 2$. Then we obtain

$$\Psi(\sigma + \zeta) - \Psi(\sigma + 1 + \delta) \ge (\zeta - 1 - \delta)\Psi'_+(\sigma + 1 + \delta) \ge 2\ln \sigma, \quad \sigma \ge \sigma_5, \tag{56}$$

with some $\sigma_5 \ge \sigma_4 - 1 - \delta$. Therefore using Theorem 1 and relations (55) and (56) for some positive $\sigma_6 \ge \sigma_5$ and all $\sigma \ge \sigma_6$ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\sigma + 1, G_c) + \ln(|a_N|/b_N) + 2\ln \sigma \le \Psi(\sigma + \zeta).$$

If $\sigma \le \sigma_2$, then using (55) we get

$$\ln \mathfrak{M}(\sigma, F) \le \ln(|a_N|/b_N) + \ln \mu(\sigma + 1, G_c) \le \Psi(\sigma + 1 + \delta).$$

If $\sigma \in [\sigma_2, \sigma_9]$, then using the monotonicity of the function Ψ we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mathfrak{M}(\sigma_6, F) \le \Psi(\sigma_6 + \zeta) \le \Psi(\sigma + \sigma_6 - \sigma_2 + \zeta).$$

As a result, we see that (53) holds with $\varepsilon = \sigma_6 - \sigma_2 + \zeta$, and Theorem 30 is proved. □

The following three theorems can be proved by similar considerations.

Theorem 31. *Let $\Psi \in X$, δ_0 be the quantity defined by* (26)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and $(b_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (11) *and* (10)*. If $\delta_0 < +\infty$, then there exist real constants ε and B such that*

$$\ln \mathfrak{M}(\sigma, F) \le \Psi(\sigma+\varepsilon) + B, \quad \sigma \in \mathbb{R}, \tag{57}$$

if and only if there exist real constants δ and C such that

$$\ln b_n \le -\widetilde{\Psi}(\lambda_n) + \delta\lambda_n + C, \quad n \in \mathbb{N}_0. \tag{58}$$

Theorem 32. *Let $\Psi \in X$, δ_0 be the quantity defined by* (26)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and $(b_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (11) *and* (10)*. If $\delta_0 < +\infty$, then for every $B > 0$ there exists a constant $\varepsilon \in \mathbb{R}$ such that* (57) *holds if and only if for every $C > \ln(|a_N|/b_N)$ there exists a constant $\delta \in \mathbb{R}$ such that* (58) *holds.*

Theorem 33. *Let $\Psi \in X$, δ_0 be the quantity defined by* (26)*, $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*, $c > 0$, and $(b_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (11) *and* (10)*. If $\delta_0 = 0$, then for every $\varepsilon > 0$ there exists a constant $B \in \mathbb{R}$ such that* (57) *holds if and only if for every $\delta > 0$ there exists a constant $C \in \mathbb{R}$ such that* (58) *holds.*

Let $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form (1). If $a_n \ge 0$ for any $n \in \mathbb{N}_0$, then $M(\sigma, F) = \mathfrak{M}(\sigma, F)$ for all $\sigma \in \mathbb{R}$, and therefore in the estimates from Theorems 28–33 we can replace $\mathfrak{M}(\sigma, F)$ with $M(\sigma, F)$. In the general situation, for all $\sigma \in \mathbb{R}$ we have $M(\sigma, F) \le \mathfrak{M}(\sigma, F)$, and therefore the mentioned theorems give only sufficient conditions for the corresponding estimates to hold for $M(\sigma, F)$. The question regarding the necessity of these conditions remains open and does not seem simple, since $\mathfrak{M}(\sigma, F)$ can grow relative to $M(\sigma, F)$ as $\sigma \to +\infty$ arbitrarily fast (see, for example, [17, 19]). However, under additional conditions on the sequence of exponents $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ of the series F, it is already possible to establish necessary and sufficient conditions, under which some general estimates from above for $M(\sigma, F)$ hold. In fact, let N be the number defined by (10) and $\tau(\lambda) = \overline{\lim}_{n\to\infty} \ln n/\lambda_n$. If $\tau(\lambda) < +\infty$, then for each $\varepsilon > \tau(\lambda)$ we have $B(\varepsilon) := \sum_{n=N}^{\infty} e^{-\varepsilon\lambda_n} < +\infty$, and therefore

$$\mu(\sigma, F) \le M(\sigma, F) \le \mathfrak{M}(\sigma, F) = \sum_{n=N}^{\infty} |a_n| e^{(\sigma+\varepsilon)\lambda_n} e^{-\varepsilon\lambda_n} \le B(\varepsilon)\mu(\sigma+\varepsilon, F), \quad \sigma \in \mathbb{R}. \tag{59}$$

Using (59) and Lemmas 8 and 9, it is easy to prove, for example, the following two theorems.

Theorem 34. *Let $\Psi \in X$, $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ such that $\tau(\lambda) < +\infty$, and $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*. Then there exist real constants ε and B such that*

$$\ln M(\sigma, F) \le \Psi(\sigma+\varepsilon) + B, \quad \sigma \in \mathbb{R}, \tag{60}$$

if and only if there exist real constants δ and C such that

$$\ln |a_n| \le -\widetilde{\Psi}(\lambda_n) + \delta\lambda_n + C, \quad n \in \mathbb{N}_0. \tag{61}$$

Theorem 35. *Let $\Psi \in X$, $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence from the class Λ such that $\tau(\lambda) = 0$, and $F \in \mathcal{D}_{+\infty}$ be a Dirichlet series of the form* (1)*. Then for every $\varepsilon > 0$ there exists a real constant B such that* (60) *holds if and only if for every $\delta > 0$ there exists a real constant C such that* (61) *holds.*

Now we turn to Dirichlet series absolutely convergent in a half-plane. Let $\Psi \in X_0$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form (1), and $(S_n)_{n\in\mathbb{N}_0}$ be the sequence defined by (4). In [23], it is proved that for every $q \in (0,1)$ there exists a constant $B \in \mathbb{R}$ such that

$$\ln \mathfrak{M}(\sigma, F) \le \Psi(q\sigma) + B, \quad \sigma < 0, \tag{62}$$

if and only if for every $p \in (0,1)$ there exists a constant $C \in \mathbb{R}$ such that

$$\ln S_n \le -\widetilde{\Psi}(\lambda_n/p) + C, \quad n \in \mathbb{N}_0, \tag{63}$$

and also it is proved that there exist constants $q \in (0,1)$ and $B \in \mathbb{R}$ such that (62) holds if and only if there exist constants $p \in (0,1)$ and $C \in \mathbb{R}$ such that (63) holds. Here we supplement these results with the following theorem.

Theorem 36. *Let $\Psi \in X_0$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, and $(S_n)_{n\in\mathbb{N}_0}$ be the sequence defined by* (4)*. Then for every $B > 0$ there exists a constant $q \in (0,1)$ such that* (62) *holds if and only if for every $C > 0$ there exists a constant $p \in (0,1)$ such that* (63) *holds.*

Proof. Let F_2 be the series defined by (5).

Necessity. Suppose that for every $B > 0$ there exists a constant $q = q(B) \in (0,1)$ such that (62) holds, and let $C > 0$. Put $p = q(B)$. Using Theorem E, for all $\sigma < 0$ we have $\ln \mu(\sigma, F_2) \le \ln \mathfrak{M}(\sigma, F) \le \Psi(p\sigma) + C$. Then by Lemmas 8 and 9 we obtain (63).

Sufficiency. Suppose that for every $C > 0$ there exists a constant $p = p(C) \in (0,1)$ such that (63) holds, and let $B > 0$. We fix some $C_0 < B$ and set $p_0 = p(C_0)$. Then we have $\ln S_n \le -\widetilde{\Psi}(\lambda_n/p_0) + C_0$ for each $n \in \mathbb{N}_0$, and therefore by Lemmas 8 and 9 for all $\sigma < 0$ we obtain $\ln \mu(\sigma, F_2) \le \Psi(p_0\sigma) + C_0$. Put $\delta = 1 - e^{C_0 - B}$. Noting that $\delta \in (0,1)$, and using Theorem E, for all $\sigma < 0$ we get

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\delta\sigma, F_2) + \ln \frac{1}{1-\delta} \le \Psi(p_0\delta\sigma) + C_0 + \ln \frac{1}{1-\delta} = \Psi(p_0\delta\sigma) + B,$$

i.e. (62) holds with $p = p_0\delta$. □

By additional assumptions regarding the growth of the function $\Psi \in X_0$, the condition on $(S_n)_{n\in\mathbb{N}_0}$ in Theorem 36 can be replaced by a simpler one.

Theorem 37. *Let $\Psi \in X_0$, $F \in \mathcal{D}_0$ be a Dirichlet series of the form* (1)*, $c > 0$, and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence defined by* (16) *and* (10)*. If*

$$\varliminf_{\sigma\uparrow 0} |\sigma|\Psi(\sigma)/\ln(1/|\sigma|) > 0, \tag{64}$$

then for every $B > 0$ there exists a constant $q \in (0,1)$ such that (62) *holds if and only if for every $C > 0$ there exists a constant $p \in (0,1)$ such that* (46) *holds.*

Proof. Let G_c be the series defined by (12).

Necessity. Suppose that for every $B > 0$ there exists a constant $q = q(B) \in (0,1)$ such that (62) holds, and let $C > 0$. We put $p = q(B)$. Using Theorem 2, for all $\sigma < 0$ we have $\ln \mu(\sigma, G_c) \le \ln \mathfrak{M}(\sigma, F) \le \Psi(p\sigma) + C$. Then by Lemmas 8 and 9 we obtain (46).

Sufficiency. Suppose that for every $C > 0$ there exists a constant $p = p(C) \in (0,1)$ such that (46) holds, and let $B > 0$. Let us fix some $C_0 < B$ and set $p_0 = p(C_0)$. Then we have $\ln b_n \le -\widetilde{\Psi}(\lambda_n/p_0) + C_0$ for each $n \in \mathbb{N}_0$, and therefore by Lemmas 8 and 9 for all $\sigma < 0$ we obtain $\ln \mu(\sigma, G_c) \le \Psi(p_0\sigma) + C_0$. Without loss of generality, we can assume that the function Ψ is nondecreasing, convex, and takes finite values on $(-\infty, 0)$ (see the proof of Theorem 30).

Let N be the number defined by (10). Since $b_N = |a_N|$, there exists $\sigma_1 < -1$ such that $\mu(\sigma, G_c) = \mu(\sigma, F)$ for all $\sigma \le \sigma_1$. Furthermore, $\mathfrak{M}(\sigma, F) \sim |a_N|e^{\sigma\lambda_N}$ as $\sigma \to -\infty$. Hence, there exists $\sigma_2 < \sigma_1$ such that

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\sigma, G_c) + B - C_0 \le \Psi(p_0\sigma) + B, \quad \sigma \le \sigma_2. \tag{65}$$

Now let us fix arbitrary $\delta \in (0,1)$ and $\delta_0 \in (0,\delta)$. Since $\Psi(\sigma) \to +\infty$ as $\sigma \uparrow 0$ by the condition (64) and Ψ is convex on $(-\infty, 0)$, we have $\Psi'_+(\sigma)/\Psi(\sigma) \to +\infty$ as $\sigma \uparrow 0$. Then from (64) it follows that $|\sigma|\Psi'_+(\sigma)/(-\ln|\sigma|) \to +\infty$ as $\sigma \uparrow 0$. Therefore, there exists $\sigma_3 \in (-1, 0)$ such that $\Psi(\delta_0 p_0\sigma) - \Psi(\delta p_0\sigma) \ge (\delta_0 - \delta)p_0\sigma\Psi'_+(\sigma) \ge -3\ln|\sigma|$, $\sigma \in (\sigma_3, 0)$. Then, using Theorem 2 (see Remark 2), for some $\sigma_4 \in (\sigma_3, 0)$ we obtain

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mu(\delta\sigma, G_c) - 2\ln|\sigma| \le \Psi(\delta p_0\sigma) - 3\ln|\sigma| < \Psi(\delta_0 p_0\sigma) + B, \quad \sigma \in [\sigma_4, 0). \tag{66}$$

Finally, if $\sigma \in [\sigma_2, \sigma_4]$, then we have

$$\ln \mathfrak{M}(\sigma, F) \le \ln \mathfrak{M}(\sigma_4, F) \le \Psi(\delta_0 p_0\sigma_4) + B \le \Psi(\delta_0 p_0|\sigma_4|\sigma/|\sigma_2|) + B.$$

This, together with (65) and (66), shows that (62) holds with $p = \delta_0 p_0|\sigma_4|/|\sigma_2|$. ☐

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Received 02.04.2024
Revised 03.03.2025

Грибель О.Б., Філевич П.В. *Оцінки для сум рядів Діріхле* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 42–66.

У роботі доведено теореми апроксимаційного характеру, що дозволяють з достатньою точністю оцінити супремум модуля ряду Діріхле через максимальний член іншого ряду Діріхле, пов'язаного зі заданим. За допомогою цих теорем для ряду Діріхле отримано умови на послідовність модулів його коефіцієнтів, які є необхідними та достатніми для виконання найзагальніших асимптотичних та глобальних оцінок зверху для його супремуму модуля.

Ключові слова і фрази: аналітична функція, ціла функція, ряд Діріхле, супремум модуля, абсциса абсолютної збіжності, максимальний член, спряжена за Юнгом функція.